BASIC MATHEMATICS USED IN PHYSICS

Mathematics is the supporting tool of Physics. Elementary knowledge of basic mathematics is useful in problem solving in Physics. In this chapter we study Elementary Algebra, Trigonometry, Coordinate Geometry and Calculus (differentiation and integration).

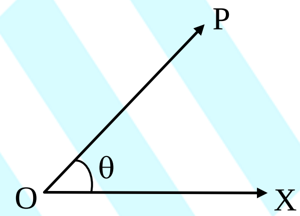
1. TRIGONOMETRY

1.1 Angle

Consider a revolving line OP.

Suppose that it revolves in anticlockwise direction starting from its initial position OX.

The angle is defined as the amount of revolution that the revolving line makes with its initial position.



From fig. the angle covered by the revolving line OP is $\theta = \angle POX$

The angle

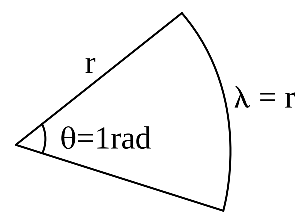
is taken **positive** if it is traced by the revolving line in anticlockwise direction and

is taken **negative** if it is covered in clockwise direction.

$$1^\circ = 60' \text{ (minute)}$$
$$1' = 60'' \text{ (second)}$$

1 right angle = 90° (degrees) also 1 right angle = $\frac{\pi}{2}$ rad (radian)

One radian is the angle subtended at the centre of a circle by an arc of the circle, whose length is equal to the radius of the circle. 1 rad = $\frac{180^\circ}{\pi} \approx 57.3^\circ$



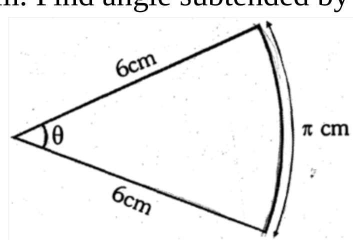
To convert an angle from degree to radian multiply it by $\frac{\pi}{180^\circ}$

To convert an angle from radian to degree multiply it by $\frac{180^\circ}{\pi}$

Illustrations

Illustration 1.

A circular arc is of length π cm. Find angle subtended by it at the centre in radian and degree.



Solution:

$$\theta = \frac{s}{r} = \frac{\pi\,\text{cm}}{6\,\text{cm}} = \frac{\pi}{6}\text{rad} = 30^\circ \quad \text{As 1 rad} = \frac{180^\circ}{\pi} \quad \text{So } \theta = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$$

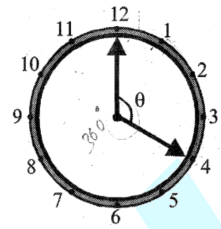
Illustration 2.

When a clock show 4 o'clock how much angle do its minute and hour needles make?

(1) 120º (2) $\frac{\pi}{3}$ rad (3) $\frac{2\pi}{3}$ rad (4) 160º

Ans. (1, 3)

Solution:



From diagram angle θ = 4 × 30º = 120º = $\frac{2\pi}{3}$ rad

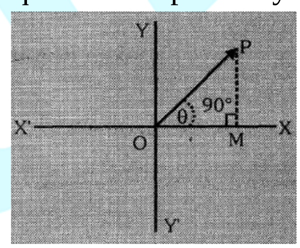
1.2 Trigonometrical ratios (or T ratios)

Let two fixed lines XOX' and YOY' intersect at right angles to each other at point O. Then,

(i) Point O is called origin.

(ii) XOX' is known as X-axis and YOY' as Y-axis.

(iii) Portions XOY, YOX, XOY' and Y'OX are called I, II, III and N quadrant respectively.



Consider that the revolving line OP has traced out angle θ (in I quadrant) in anticlockwise direction. From P, draw perpendicular PM on OX. Then, side OP (in front of right angle) is called hypotenuse, side MP (in front of angle θ) is called **opposite side or perpendicular** and side OM (making angle θ with hypotenuse) is called **adjacent side or base**.

The three sides of a right angled triangle are connected to each other through six different ratios, called trigonometric ratios or simply T -ratios :

$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{MP}{OP}$ $\cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP}$

$\text{ten}\,\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{MP}{OM}$ $\cot\theta = \frac{\text{base}}{\text{perpendicular}} = \frac{OM}{MP}$

$\sec\theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM}$ $\text{coses}\,\theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{MP}$

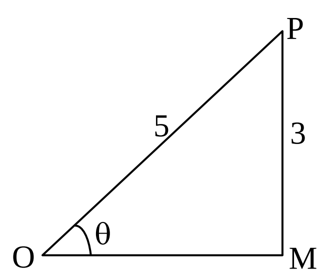
It can be easily proved that :

$\text{coses}\,\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

$\sin^2\theta + \cos^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \text{coses}^2\theta$

Illustration 3.

Given sin θ = 3/5. Find all the other T-ratios, if θ lies in the first quadrant.



In Δ OMP, $\sin\theta = \frac{3}{5}$ so MP = 3 and OP = 5

Θ $OM = \sqrt{(5)^2 - (3)^2} = \sqrt{25-9} = \sqrt{16} = 4$

Now, $\cos\theta = \frac{OM}{OP} = \frac{4}{5}$ $\tan\theta = \frac{MP}{OM} = \frac{3}{4}$

$\text{Cot}\,\theta = \frac{OM}{MP} = \frac{4}{3}$ $\sec\theta = \frac{OP}{OM} = \frac{5}{4}$ $\text{coses}\,\theta = \frac{OP}{MP} = \frac{5}{3}$

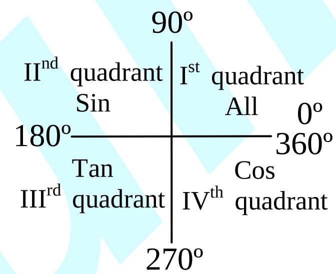
Table : The T- ratios of a few standard angles ranging from 0º to 180º

Angle (θ)	0º	30º	45º	60º	90º	120º	135º	150º	180º
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ (not defined)	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0

1.3 Four Quadrants and ASTC Rule*

In first quadrant, all trigonometric ratios are positive.
In second quadrant, only sinθ and cosecθ are positive.
In third quadrant, only tanθ and cotθ are positive.
In fourth quadrant, only cosθ and secθ are positive

* Remember as Add Sugar To Coffee or After School To College.



1.4 Trigonometrical Ratios of General Angles (Reduction Formulae)

(i) Trigonometric function of an angle (2nπ+θ) where n=0, 1, 2, 3,... will be remain same.

sin(2nπ+θ) = sinθ cos(2nπ+θ) = cosθ tan(2nπ+θ) = tanθ

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will remain same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

sin(π–θ) = +sinθ cos(π–θ) = –cosθ tan(π–θ) = –tanθ
sin(π+θ) = –sinθ cos(π+θ) = –cosθ tan(π+θ) = +tanθ
sin(2π–θ) = –sinθ cos(2π–θ) = +cosθ tan(2π–θ) = –tanθ

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$\sin\left(\frac{\pi}{2}+\theta\right) = +\cos\theta$ $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$ $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$ $\cos\left(\frac{\pi}{2}-\theta\right) = +\sin\theta$ $\tan\left(\frac{\pi}{2}-\theta\right) = +\cot\theta$

(iv) Trigonometric function of an angle –θ (negative angles)

sin(–θ) = –sinθ cos(–θ) = +cosθ tan(–θ) = tanθ

sin (90° + θ) = cosθ cos (90° + θ) = –sinθ tan (90° + θ) = –cotθ	sin (180° – θ) = sinθ cos (180° – θ) = –cosθ tan (180° – θ) = –tanθ	sin (–θ) = –sinθ cos (–θ) = cosθ tan (–θ) = –tanθ	sin (90° – θ) = cosθ cos (90° – θ) = sinθ tan (90° – θ) = cotθ
sin (180° + θ) = –sinθ cos (180° + θ) = –cosθ tan (180° + θ) = tanθ	sin (270° – θ) = –cosθ cos (270° – θ) = –sinθ tan (270° – θ) = cotθ	sin (270° + θ) = –cosθ cos (270° + θ) = sinθ tan (270° + θ) = –cotθ	sin (360° – θ) = –sinθ cos (360° – θ) = cosθ tan (360° – θ) = –tanθ

Illustration 4.

Find the value of

(i) cos (–60º) (ii) tan 210º (iii) sin 300º (iv) cos 120º

Solution:

(i) $\cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}$

(ii) $\tan 210^\circ = \tan(180^\circ + 30^\circ) = \tan 30^\circ \; \frac{1}{\sqrt{3}}$

(iii) $\sin 300^\circ = \sin(270^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

(iv) $\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

BEGINNER'S BOX - 1

1. Find the values of :

(i) tan (–30º) (ii) sin 120º (iii) sin 135º (iv) cos 150º

(v) sin 270º (vi) cos 270º

2. If $\sec\theta = \frac{5}{3}$ and $0 < \theta < \frac{\pi}{2}$. Find all the other T-ratios.

1.5 A few important trigonometric formulae

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

$\cos 2A = 2\cos^2 A - 1 - 2\sin^2 A$

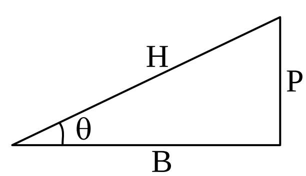
$1 + \cos A = 2\cos^2 \frac{A}{2}$, $1 - \cos A = 2\sin^2 \frac{A}{2}$

1.6 Range of trigonometric functions

As $\sin\theta = \frac{P}{H}$ and $P \le H$ so $-1 \le \sin\theta \le 1$

As $\cos\theta = \frac{B}{H}$ and $B \le H$ so $-1 \le \cos\theta \le 1$

As $\tan\theta = \frac{P}{B}$ so $-\infty < \tan\theta < \infty$



Remember : $-\sqrt{a^2 + b^2} \le a \sin\theta + b \cos\theta \le \sqrt{a^2 + b^2}$

1.7 Small Angle Approximation

If θ is small (say < 5º) then $\sin\theta \approx \theta$, $\cos\theta \approx 1$ & $\tan\theta \approx \theta$. Here θ must be in radians.

Illustration 5.

Find the approximate values of (i) sin 1º (ii) tan 2º (iii) cos 1º

Solution:

(i) $\sin 1^\circ = \sin\left(1^\circ \times \frac{\pi}{180^\circ}\right) = \sin\left(\frac{\pi}{180}\right) \approx \frac{\pi}{180}$

(ii) $\tan 2^\circ = \tan\left(2^\circ \times \frac{\pi}{180^\circ}\right) = \tan\left(\frac{\pi}{90}\right) \approx \frac{\pi}{90}$

(iii) $\cos 1^\circ \approx 1$

2. COORDINATE GEOMETRY

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If a point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three, coordinates are needed.

- **Origin**

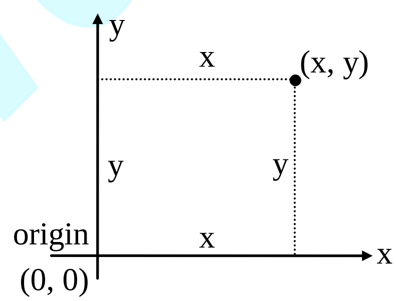
  This is any fixed point which is convenient to you. All measurements are taken w.r.t. this fixed point.

- **Axis or Axes**

  Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.

2.1 Position of a point in xy plane

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure. Here x-coordinate and y-coordinate is called abscissa and ordinate respectively.



2.2 Distance Formula

The distance between two point (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1^{\,2})}$

Note : In space $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

2.3 Slop of a Line

The slope of a line joining two points A(x_1, y_1) and B(x_2, y_2) is denoted by m and is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta \text{ [If both axes have identical scales]}$$

Here θ is the angle made by line with positive x-axis.
Slope of a line is a quantitative measure of inclination.

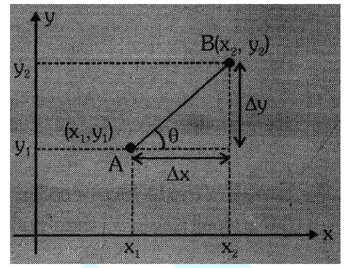


Illustration 6.

For point (2, 14) find abscissa and ordinate. Also distance from y and x-axis.

Solution:

Abscissa = x-coordinate = 2 = distance from y-axis.
Ordinate = y-coordinate = 14 = distance from x=axis.

Illustration 7.

Find value of a if distance between the points (–9 cm, a cm) and (3 cm, 3 cm) is 13 cm.

Solution:

By using distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1{}^2)} \Rightarrow 13 = \sqrt{[3-(-9)]^2 + [3 - a^2]}$

$\Rightarrow 13^2 = 12^2 + (3–a)^2 \Rightarrow (3–a)^2 = 13^2 – 12^2 = 5^2 \Rightarrow (3–a) = \pm 5 \Rightarrow a = –2$ cm or 8 cm

Illustration 8.

A dog wants to catch a cat. The dog follows the path whose equation is $y – x = 0$ while the cat follow the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are :

Ans. **(2, 4)**

Solution:

Let catching point be (x_1, y_1) then $y_1 – x_1 = 0$ and $x_1^2 + y_1^2 = 8$

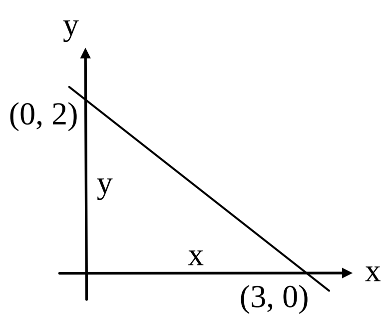
Therefore, $2x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$; so possible points are (2, 2) and (–2, –2).

BEGINNER'S BOX - 2

1. Distance between two points (8, –4) and (0, a) is 10. All the values are in the same unit of length. Find the positive value of a.

2. Calculate the distance between two points (0, –1, 1) and (3, 3, 13).

3. Calculate slope of shown line



3. DIFFERENTIATION

3.1 Function

Constant: A quantity, whose value remains unchanged during mathematical operations, is called a constant quantity. The integers, fractions like π, e etc are all constants.

Variable: A quantity, which can take different values, is called a variable quantity. A variable is usually represented as x, y, z, etc.

Function: A quantity y is called a function of a variable x, if corresponding to any given value of x, there exists a single definite value of y. The phrase 'y is function of x' is represented as y = f (x)

For example, consider that y is a function of the variable x which is given by $y = 3x^2 + 7x + 2$

If x = 1, then $y = 3(1)^2 + 7(1) + 2 = 12$ and when x = 2, $y = 3(2)^2 + 7(2) + 2 = 28$

Therefore, when the value of variable x is changed, the value of the function y also changes but corresponding to each value of x, we get a single definite value of y. Hence, $y = 3x^2 + 7x + 2$ represents a function of x.

3.2 Physical meaning of $\frac{dy}{dx}$

(i) The ratio of small change in the function y and the variable x is called the average rate of change of y w.r.t. x.

For example, if a body covers a small distance Δs in small time Δt, then

average velocity of the body $v_{av} = \frac{\Delta s}{\Delta t}$

Also, if the velocity of a body changes by a small amount Δv in small time Δt, then

average acceleration of the body, $a_{av} = \frac{\Delta v}{\Delta t}$

(ii) When $\Delta x \to 0$ the limiting value of $\frac{\Delta y}{\Delta x}$ is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

It is called the instantaneous rate of change of y w.r.t. x.

The differentiation of a function w.r.t. a variable implies the instantaneous rate of change of the function w.r.t. that variable.

Like wise, instantaneous velocity of the body $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$

and instantaneous acceleration of the body $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$

3.3 Theorems of differentiation

1. If c = constant, $\frac{d}{dx}(c) = 0$

2. y = c u, where c is a constant and u is a function of x, $\frac{dy}{dx} = \frac{d}{dx}(cu) = c\frac{du}{dx}$

3. y = u ± v ± w, where, u, v and w are functions of x, $\frac{dy}{dx} = \frac{d}{dx}(u \pm v \pm w) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx}$

4. y = u where u and v are functions of x, $\frac{dy}{dx} = \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

5. $y = \frac{u}{v}$, where u and v are functions of x, $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

6. $y = x^n$, n real number, $\frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$

Illustration 9.

Find $\frac{dy}{dx}$, when (i) $y = \sqrt{x}$ (ii) $y = x^5 + x^4 + 7$ (iii) $y = x^2 + 4x^{-1/2} - 3x^{-2}$

Solution:

(i) $y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

(ii) $y = x^5 + x^4 + 7 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^5 + x^4 + 7) = \frac{d}{dx}(x^5) + \frac{d}{dx}(x^4) + \frac{d}{dx}(7)$

$= 5x^4 + 4x^3 + 0 = 5x^4 + 4x^3$

(iii) $y = x^2 + 4x^{-1/2} - 3x^{-2} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 + 4x^{-1/2} - 3x^{-2}) = \frac{d}{dx}(x^2) + \frac{d}{dx}(4x^{-1/2}) - \frac{d}{dx}(3x^{-2})$

$= \frac{d}{dx}(x^2) + 4\frac{d}{dx}(x^{-1/2}) - 3\frac{d}{dx}(x^{-2}) = 2x + 4\left(-\frac{1}{2}\right)x^{-3/2} - 3(-2)x^{-3}$

$= 2x - 2x^{-3/2} + 6x^{-3}$

BEGINNER'S BOX - 3

1. Find $\frac{dy}{dx}$ for the following :

(i) $y = x^{7/2}$

(ii) $y = x^{-3}$

(iii) $y = x$

(iv) $y = x^5 + x^3 + 4x^{1/2} + 7$

(v) $y = 5x^4 + 6x^{3/2} + 9x$

(vi) $y = ax^2 + bx + c$

(vii) $y = 3x^5 - 3x - \frac{1}{x}$

2. Given $s = t^2 + 5t + 3$, find $\frac{ds}{dt}$.

3. If $s = ut + \frac{1}{2}at^2$, where u and a are constants. Obtain the value of $\frac{ds}{dt}$.

4. The area of a blot of ink is growing such that after t seconds, its area is given by $A = (3t^2 + 7)$ cm^2. Calculate the rate of increase of area at t = 5 second.

5. The area of a circle is given by $A = \pi r^2$, where r is the radius. Calculate the rate of increase of area w.r.t. radius.

6. Obtain the differential coefficient of the following :

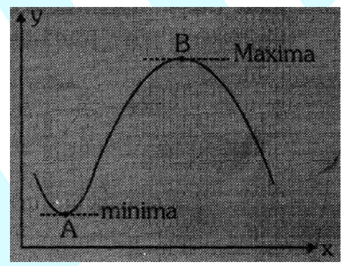
(i) (x – 1) (2x + 5) (ii) $\frac{1}{2x+1}$ (iii) $\frac{3x+4}{4x+5}$ (iv) $\frac{x^2}{x^3+1}$

3.4 Formulae for differential coefficients of trigonometric, logarithmic and exponential functions

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
- $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$

3.5 Maximum and Minimum value of a Function

Higher order derivatives are used to find the maximum and minimum values of a function. At the point of maxima and minima, first derivative (i.e. $\frac{dy}{dx}$) becomes zero.



At point 'A' (minima) :
As we see in figure, in the neighbourhood of A, slope increases so $\frac{d^2y}{dx^2} > 0$.

Condition for minima : $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

At point 'B' (maxima) : As we see in figure, in the neighborhood of B, slope decreases so $\frac{d^2y}{dx^2} < 0$.

Condition for maxima : $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

Illustration 10.
The minimum value of $y = 5x^2 - 2x + 1$ is

(1) $\frac{1}{5}$ (2) $\frac{2}{5}$ (3) $\frac{4}{5}$ (4) $\frac{3}{5}$

Ans. (3)
Solution:

For maximum/minimum value $\frac{dy}{dx} = 0 \Rightarrow 5(2x) - 2(1) + 0 = 0 \Rightarrow x = \frac{1}{5}$.

Now at $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = 10$ which is positive

so y has minimum value at $x = \frac{1}{5}$. Therefore, $y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 = \frac{4}{5}$

4. INTEGRATION

In integral calculus, the differential coefficient of a function is given. We are required to find the function. Integration is basically used for summation. Σ is used for summation of discrete values, while $\int$ sign is used for continuous function.

If I is integration of f(x) with respect to x then

$I = \int f(x)\,dx$ [we can check $\frac{dI}{dx} = f(x)$] $\because \int f'(x)\,dx = f(x) + c$

where c = an arbitrary constant

Let us proceed to obtain intergral of x^n w.r.t. x. $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$

Since the process of integration is the reverse process of differentiation,

$$\int (n+1)\,x^n\,dx = x^{n+1} \quad \text{or} \quad (n+1)\int x^n dx = x^{n+1} \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1}$$

The above formula holds for all values of n, except n = –1.

It is because, for n = –1, $\int x^n dx = \int x^{-1} dx = \int \frac{1}{x} dx$

$$\because \quad \frac{d}{dx}(\log_e x) = \frac{1}{x} \qquad \therefore \quad \int \frac{1}{x} dx = \log_e x$$

Similarly, the formulae for integration of some other functions can be obtained if we know the differential coefficients of various functions.

4.1 Few basic formulae of integration

Following are a few basic formulae of integration :

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, Provided $n \neq -1$
2. $\int \sin x\,dx = -\cos x + c$ $\quad (\because \frac{d}{dx}(\cos x) = -\sin x)$
3. $\int \cos x\,dx = \sin x + c$ $\quad (\because \frac{d}{dx}(\sin x) = \cos x)$
4. $\int \frac{1}{x}\,dx = \log_e x + c$ $\quad (\because \frac{d}{dx}(\log_e x) = \frac{1}{x})$
5. $\int e^x\,dx = e^x + c$ $\quad (\because \frac{d}{dx}(e^x) = e^x)$

Illustration 11.

Integrate w.r.t. x. : (i) $x^{11/2}$ (ii) x^{-7} (iii) $x^{p/q}(p/q \neq -1)$

Solution:

(i) $$\int x^{11/2} dx = \frac{x^{11/2+1}}{\frac{11}{2}+1} + c = \frac{2}{13}x^{13/2} + c$$

(ii) $$\int x^{-7} dx = \frac{x^{-7+1}}{-7+1} + c = -\frac{1}{6}x^{-6} + c$$

(iii) $\int x^{\frac{p}{q}}dx = \dfrac{x^{\frac{p}{q}+1}}{\frac{p}{q}+1} + c = \dfrac{q}{p+q}x^{(p+q)} + c$

Illustration 12.

Evaluate $\int \left(x^2 - \cos x + \dfrac{1}{x} \right) dx$

Solution:

$I = \int x^2 dx - \int \cos x dx + \int \dfrac{1}{x} dx = \dfrac{x^{2+1}}{2+1} - \sin x + \log_e x + c = \dfrac{x^3}{3} - \sin x + \log_e x + c$

BEGINNER'S BOX - 4

1. Evaluate the following integrals :

(i) $\int x^{15}dx$ (ii) $\int x^{-3/2}dx$ (iii) $\int (3x^{-7} + x^{-1})dx$ (iv) $\int \left(\sqrt{x} + \dfrac{1}{\sqrt{x}} \right)^2 dx$

(v) $\int \left(x + \dfrac{1}{x} \right) dx$ (vi) $\int \left(\dfrac{a}{x^2} + \dfrac{b}{x} \right) dx$ (a and b are constant)

4.2 Definite Integrals

When a function is integrated between a lower limit and an upper limit, it is called a definite integral.

If $\dfrac{d}{dx}(f(x)) = f'(x)$, then

$\int f'(x)\,dx$ is called indefinite integral and $\int_a^b f'(x)\,dx$ is called definite integral

Here, a and b are called lower and upper: limits of the variable x.

After carrying out integration, the result is evaluated between upper and lower limits as explained below :

$$\int_a^b f'(x)\,dx = |f(x)|_a^b = f(b) - f(a)$$

4.3 Area under a curve and definite integration

Area of small shown darkly shaded element = ydx = f(x) dx

If we sum up all areas between x = a and x = b then

$\int_a^b f(x)\,dx$ = shaded area between curve and x-axis.

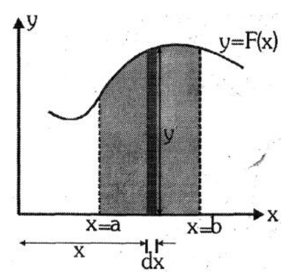


Illustration 13.

The integral $\int_1^5 x^2 dx$ is equal to

(1) $\frac{125}{3}$ (2) $\frac{124}{3}$ (3) $\frac{1}{3}$ (4) 45

Ans. (2)

Solution:

$$\int_1^5 x^2dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3}-\frac{1^3}{3}\right] = \frac{125}{3}-\frac{1}{3} = \frac{124}{3}$$

BEGINNER'S BOX - 5

1. Evaluate the following integrals :

(i) $\int_R^\infty \frac{GMm}{x^2}dx$ (ii) $\int_{r_1}^{r_2} -k\frac{q_1q_2}{x^2}dx$ (iii) $\int_u^v Mvdv$ (iv) $\int_0^\infty x^{-1/2}dx$

(v) $\int_0^{\pi/2} \sin x\,dx$ (vi) $\int_0^{\pi/2} \cos x\,dx$ (vii) $\int_{-\pi/2}^{\pi/2} \cos x\,dx$

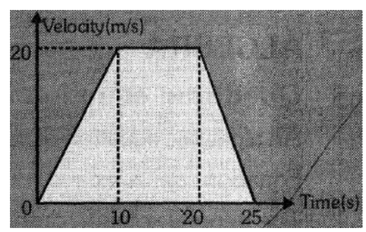
4.4 Average value of a continuous function in an interval

Average value of a function y = f(x), over an interval $a \le x \le b$ is given by

$$y_{av} = \frac{\int_a^b ydx}{\int_a^b dx} = \frac{\int_a^b ydx}{b-a}$$

Illustration 14.

The velocity-time graph of a car moving along a straight road is shown in figure. The average velocity of the car in first 25 seconds is



(1) 20 m/s (2) 14 m/s
(3) 10 m/s (4) 17.5 m/s

Ans. (2)

Solution:

$$\text{Average velocity} = \frac{\int_0^{25} vdt}{25-0} = \frac{\text{Area of } v - t \text{ graph between } t = 0 \text{ to } t = 25\text{ s}}{25} = \frac{1}{25}\left[\left(\frac{25+10}{2}\right)(20)\right] = 14\text{ m/s}$$

Illustration 15.

Determine the average value of y = 2x + 3 in the interval $0 \le x \le 1$.

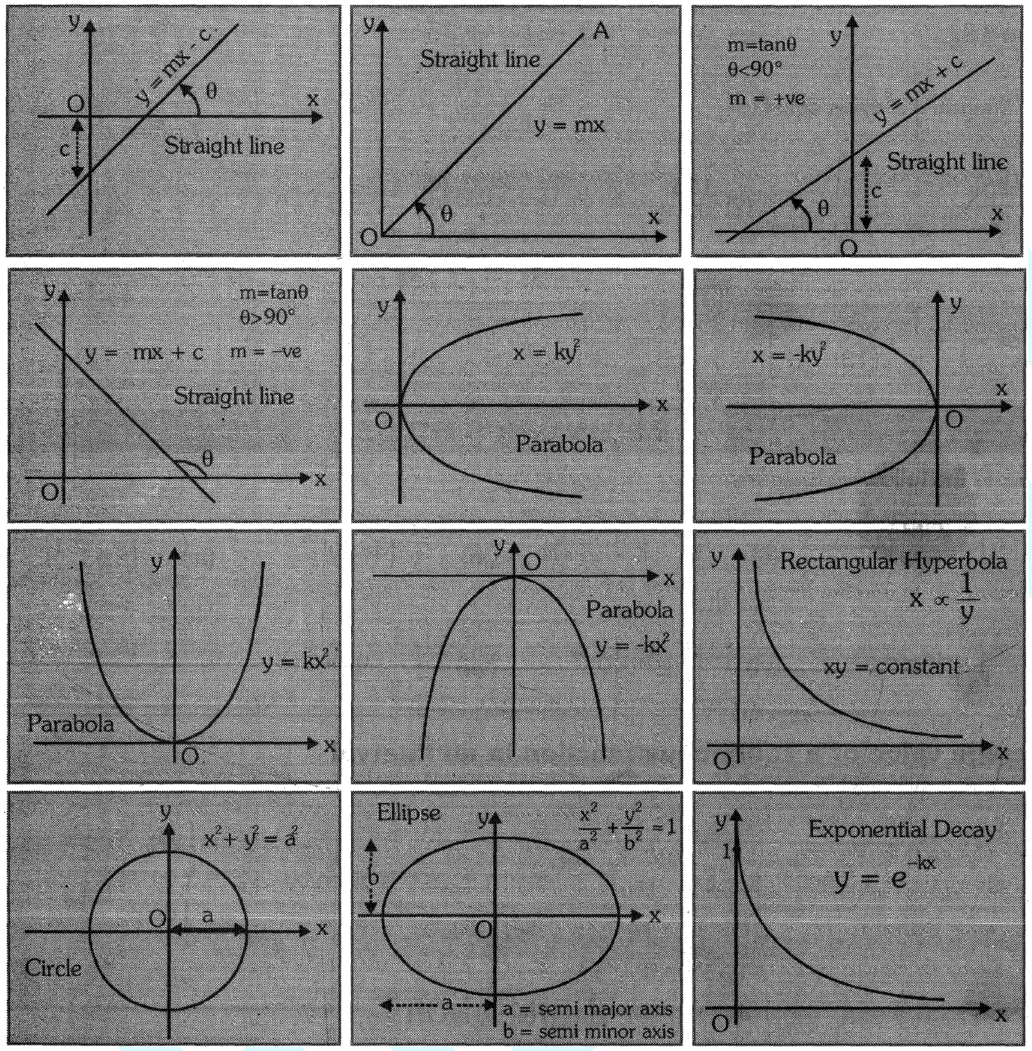
(1) 1 (2) 5 (3) 3 (4) 4

Ans. (4)

Solution:

$$y_{av} = \frac{\int_0^1 ydx}{1-0} = \int_0^1 (2x+3dx) = \left[2\left(\frac{x^2}{2}\right)+3x\right]_0^1 = 1^2 + 3(1) - 0^2 - 3(0) = 1 + 3 = 4$$

5. SOME STANDARD GRAPHS AND THEIR EQUATIONS



6. ALGEBRA

6.1 Quadratic equation and its solution :

An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation. Equation $ax^2 + bx + c = 0$ is the general quadratic equation.

The general solution of the above quadratic equation or value of variable x is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Sum of roots = $x_1 + x_2 = \frac{b}{a}$ and product of roots = $x_2 x_2 = \frac{c}{a}$

For real roots discriminant $b^2 - 4ac \geq 0$ and for imaginary roots $b^2 - 4ac < 0$

Illustration 16.

Solve the equation $2x^2 + 5x - 12 = 0$

Solution:

By comparison with the standard quadratic equation $a = 2, b = 5$ and $c = -12$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4 \times 2 \times (-12)}}{2 \times 2} = \frac{-5 \pm \sqrt{121}}{4} = \frac{-5 \pm 11}{4} = \frac{+6}{4}, \frac{-16}{4} \text{ or } \quad x = \frac{3}{2}, -4$$

Illustration 17.

The speed (v) of a particle moving along a straight line is given by $v = t^2 + 3t - 4$ where v is in m/s and t in seconds. Find time t at which the particle will momentarily come to rest.

Solution:

When particle comes to rest, v = 0

So $\quad t^2 + 3t - 4 = 0 \Rightarrow \quad t = \dfrac{-3 \pm \sqrt{9 - 4(1)(-4)}}{2(1)} \quad \Rightarrow \quad t = 1 \text{ or } -4$

Neglect negative value of t, Hence t = 1 s

Illustration 18.

The speed (v) and time (t) for an object moving along straight line are related as $t^2 + 100 = 2vt$ where v is in meter/second and t is in second. Find the possible positive values of v.

Solution:

For possible values of v, time t must be real so from $b^2 - 4ac \geq 0$

We have $\quad (-2v)^2 - 4(1)(100) \geq 0$

$\Rightarrow \quad 4v^2 - 400 \geq 0 \quad \Rightarrow \quad v^2 - 100 \geq 0$

$\Rightarrow \quad (v - 10)(v + 10) \geq 0 \quad \Rightarrow \quad v \geq 10 \text{ and } v \leq -10$

Hence possible positive values of v are v ≥ 10 m/s.

BEGINNER'S BOX - 6

1. Solve for x : (i) $10x^2 - 27x + 5 = 0$ (ii) $pqx^2 - (p^2 + q^2)x + pq = 0$

2. In quadratic equation $ax^2 + bx + c = 0$, if discriminant is $D = b^2 - 4ac$, then roots of the quadratic equation are :

(l) Real and distinct, if D > 0
(2) Real and equal (i.e., repeated roots), if D = 0.
(3) Non-real (i.e. imaginary), if D < 0
(4) All of the above are correct

6.2 Binomial Expression :

An algebraic expression containing two terms is called a binomial expression.

For example (a + b), $(a + b)^3$, $(2x - 3y)^{-1}$, $\left(x + \dfrac{1}{y}\right)$ etc. are binomial expressions.

Binomial Theorem

$(a + b^n)\ a^n + na^{n-1}b^1 + \dfrac{n(n-1)}{2 \times 1} a^{n-2}b^2 + \ldots,$ $\qquad (1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2 \times 1} x^2 + \ldots$

Binomial Approximation

If x is very small, compared to 1, then terms containing higher powers of x can be neglected so $(1 + x)^n \approx 1 + nx$

Illustration 19.

Calculate $\sqrt{0.99}$

Solution:

$$\sqrt{0.99} = (1 - 0.01)^{1/2} \approx 1 - \frac{1}{2}(0.01) \approx 1 - 0.005 \approx 0.995$$

Illustration 20.

The mass m of a body moving with a velocity v is given by $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ where m_0 = rest mass of body = 10 kg and c = speed of light = 3×10^8 m/s. find the value of m at v = 3×10^7 m/s.

Solution:

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 10\left[1-\left(\frac{3\times10^7}{3\times10^8}\right)^2\right]^{-1/2} = 10\left[1-\frac{1}{100}\right]^{-1/2} \approx 10\left[1-\left(-\frac{1}{2}\right)\left(\frac{1}{100}\right)\right] = 10 + \frac{10}{200} \approx 10.05 \text{ kg}$$

6.3 Logarithm

Common formulae :

- $\log mn = \log m + \log n$
- $\log \dfrac{m}{n} = \log m - \log n$
- $\log m^n = n \log m$
- $\log_e m = 2.303 \log_{10} m$

6.4 Componen do and Dividendo Rule : If $\dfrac{p}{q} = \dfrac{a}{b}$ then $\dfrac{p+q}{p-q} = \dfrac{a+b}{a-b}$

6.5 Arithmetic progression (AP)

General form : a, a + d, a + 2d, …., a + (n – 1) d. Here a = first term, d = common difference

Sum of n terms $S_n = \dfrac{n}{2}[a + a + (n-1)d] = \dfrac{n}{2}[2a + (n-1)d] = \dfrac{n}{2}[\text{I}^{\text{st}} \text{ term} + n^{\text{th}} \text{ term}]$

Illustration 21.

Find the sum of given Arithmetic Progression 4 + 8 + 12 +……. + 64

(1) 464 (2) 540 (3) 544 (4) 646

Ans. **(3)**

Solution:

Here a = 4, d = 4, n = 16 So, sum = $\dfrac{n}{2}$[First term + last term] = $\dfrac{16}{2}[4 + 64] = 8(68) = 544$

Note :

(i) Sum of first n natural numbers $S_n = 1 + 2 + 3 + \ldots + n = \dfrac{n}{2}[1 + n] = \dfrac{n(n+1)}{2}$

(ii) Sum of square of first n natural numbers $S_{n^2} = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) Sum of cube of first n natural numbers $S_{n^3} = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

BEGINNER'S BOX - 7

1. Find sum of first 50 natural numbers.

2. Find $1^2 + 2^2 + \ldots\ldots\ldots\ldots + 10^2$.

6.6 Geometric Progression (GP)

General form : a, ar, ar^2,….., ar^{n-1} Here a = first term, r = common ratio

Sum of n terms $S_n = \dfrac{a(1-r^n)}{1-r}$ For $0 \le |r| < 1$ Sum of ∞ term $S_\infty = \dfrac{a}{1-r}$ $(\because r<1 \therefore r^\infty \to 0)$

Illustration 22.

Find the sum of given series 1 + 2 + 4 + 8 +……. + 256

(1) 510   (2) 511   (3) 512   (4) 513

Ans. (2)

Solution:

Here a = 1, r = 2, n = 9 $(\because 256 = 2^8)$. So, $S_9 = \dfrac{(1)(1-2^9)}{(1-2)} = 2^9 - 1 = 512 - 1 = 511$

Illustration 23.

Find $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots$.. upto ∞.

(1) ∞   (2) 1   (3) 2   (4) 1.925

Ans. (3)

Solution:

Here, a = 1, r = $\dfrac{1}{2}$ So, $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{2}} = 2$

BEGINNER'S BOX - 8

1. Find $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \dfrac{1}{16} - \dfrac{1}{32} + \ldots$.... upto ∞.

2. Find $F_{net} = GMm\left[\dfrac{1}{r^2} + \dfrac{1}{2r^2} + \dfrac{1}{4r^2} + \ldots\ldots \text{upto } \infty\right]$.

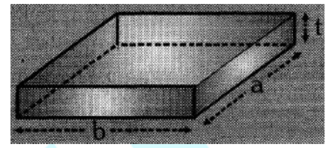
7. GEOMETRY

7.1 Formulae for determination of area :

1. Area of a square = $(\text{side})^2$
2. Area of rectangle = length × breadth
3. Area of a triangle = $\dfrac{1}{2}$ (base × height)
4. Area of trapezoid = $\dfrac{1}{2}$ (distance between parallel sides) × (sum of parallel sides)
5. Area enclosed by a circle = πr^2 (r = radius)
6. Surface area of a sphere = $4\pi r^2$ (r = radius)
7. Area of a parallelogram = base × height
8. Area of curved surface of cylinder = $2\pi r\lambda$ (r = radius and λ = length)
9. Area of ellipse = π ab   (a and b are semi major and semi minor axes respectively)
10. Surface area of a cube = $6(\text{side})^2$

11. Total surface area of cone = $\pi r^2 + \pi r\lambda$ where $\pi r\lambda = \pi r\sqrt{r^2 + h^2}$ = lateral area

7.2 Formulae for determination of volume :

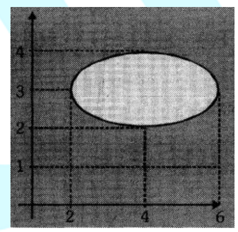


1. Volume of a rectangular slab = length × breadth × height = abt
2. Volume of a cube = $(\text{side})^3$
3. Volume of a sphere = $\frac{4}{3}\pi r^3$ (r = radius)
4. Volume of a cylinder = $\pi r^2\lambda$ (r = radius and λ is length)
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$ (r = radius and h is height)

Note : $\pi = \frac{22}{7} = 3.14$; $\pi^2 = 9.8776 \approx 10$ and $\frac{1}{\pi} = 0.3182 \approx 0.3$.

Illustration 24.

Calculate the area enclosed by shown ellipse



Solution:

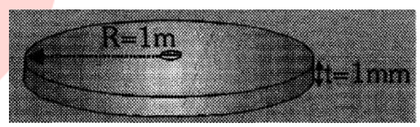
Shaded area = Area of ellipse = πab

Here $a = 6 - 4 = 2$ and $b = 4 - 3 = 1$

⇒ Area = p × 2 × 1 = 2π units

Illustration 25.

Calculate the volume of given disk.



Solution:

Volume = $\pi R^2 t = (3.14)(1)^2(10^{-3}) = 3.14 \times 10^{-3}$ m^2

VECTORS

Scalar Quantities

A physical quantity which can be described completely by its magnitude and does not require a direction is known as a scalar quantity.

It obeys the ordinary rules of algebra.

Ex. : Distance, mass, time, speed, density, volume, temperature, electric current etc.

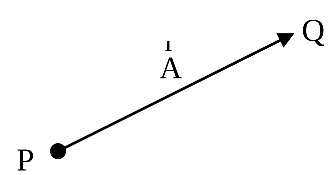
Vector Quantities

A physical quantity which requires magnitude and a particular direction, when it is expressed.

Ex.: Displacement, velocity, acceleration, force etc.

A vector is represented by a line headed with an arrow.

A vector is represented by a line headed with an arrow.



Its length is proportional to its magnitude.

$\vec{A}$ is a vector.

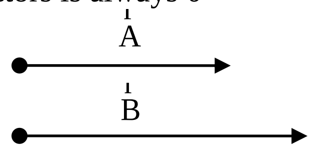
$\vec{A} = \vec{PQ}$

Magnitude of $\vec{A} = |\vec{A}|$ or A

1.1 Type of vector

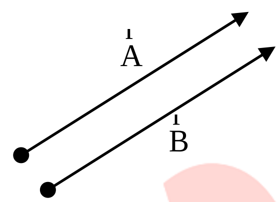
- **Parallel Vectors :-**

  Those vectors which have same direction are called parallel vectors.
  Angle between two parallel vectors is always 0º



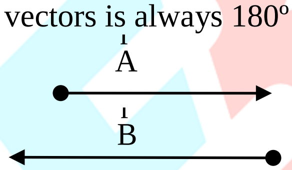
- **Equal Vectors**

  Vectors which have equal magnitude and same direction are called equal vectors.



- **Anti-parallel Vectors :**

  Those vectors which have opposite direction are called anti-parallel vector.
  Angle between two anti-parallel vectors is always 180º



- **Negative (or Opposite) Vectors**

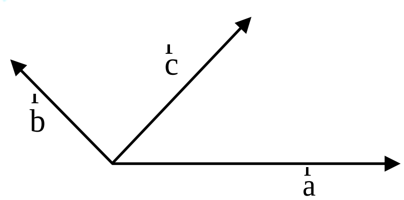
  Vectors which have equal magnitude but opposite direction are called negative vectors of each other.

  $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are negative vectors

  $\overrightarrow{AB} = -\overrightarrow{BA}$

- **Co-initial vector**

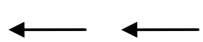
  Co-initial vectors are those vectors which have the same initial point.
  In figure $\vec{a}, \vec{b}$ and $\vec{c}$ are co-initial vectors.



- **Collinear Vectors :**

  The vectors lying in the same line are known as collinear vectors.
  Angle between collinear vectors is either 0º or 180º.

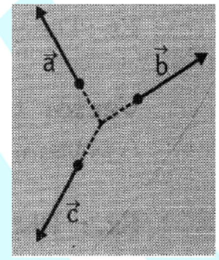
Example.

(i)  (θ = 0º) (ii)  (θ = 0º)

(iii) $\longleftarrow \quad \longrightarrow$ $(\theta = 180^\circ)$ (iv) $\longrightarrow \quad \longleftarrow$ $(\theta = 180^\circ)$

- **Coplanar Vectors**
  Vectors located in the same plane are called coplanar vectors.
  Note :- Two vectors are always coplanar.

- **Concurrent Vectors**
  Those vectors which pass through a common point are called concurrent vectors.
  In figure $\vec{a}, \vec{b}$ and $\vec{c}$ are concurrent vectors.

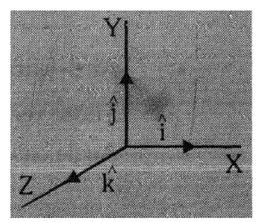


- **Null or Zero Vector**
  A vector having zero magnitude is called null vector.
  Note : Sum of two vectors is always a vector so, $(\vec{A}) + (-\vec{A}) = \vec{0}$
  $\vec{0}$ is a zero vector or null vector.

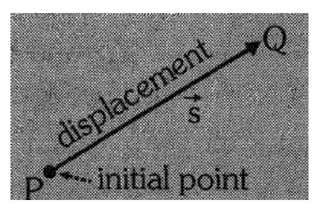
- **Unit Vector**
  A vector having unit magnitude is called unit vector. It is used to specify direction. A unit vector is represented by $\hat{A}$ (Read as A cap or A hat or A caret).
  Unit vector in the direction of $\vec{A}$ is
  $$\hat{A} = \frac{\vec{A}}{|\vec{A}|} \left(\text{unit vector} = \frac{\text{Vector}}{\text{Magnitude of the vector}} \right)$$
  $$\vec{A} = A\hat{A} = |\vec{A}|\hat{A}$$
  A unit vector is used to specify the direction of a vector.

Base Vectors

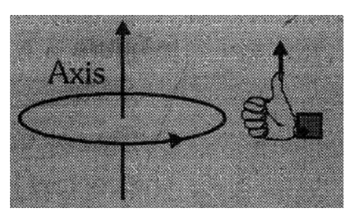
In an XYZ co-ordinate frame there are three unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, these are used to indicate X, Y and Z directions respectively. These three unit vectors are mutually perpendicular to each other.



- **Polar Vector**
  Vectors which have initial point or a point of application are called polar vectors.
  Ex. : Displacement, force etc.



- **Axial Vector**
  These vectors are used in rotational motion to define rotational effects. Direction of these vectors is always along the axis of rotation in accordance with right hand screw rule or right hand thumb rule.
  Ex. : Infinitesimal angular displacement $(\vec{d\theta})$, Angular velocity



$(\vec{\omega})$, Anglular momentum $(\vec{J})$, Angular acceleration $(\vec{\alpha})$ and Torque $(\vec{\tau})$

1.2 Addition of two vectors

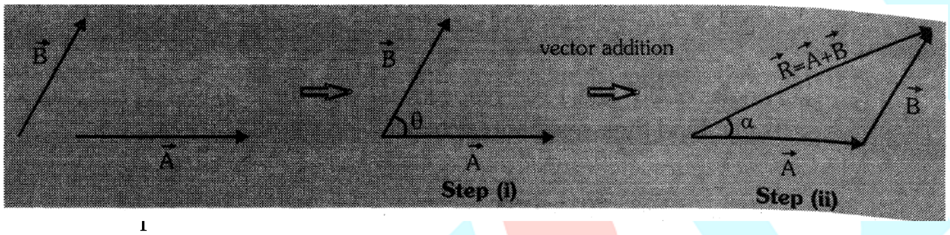
Vector addition can be performed by using following methods

(i) Graphical methods (ii) Analytical methods

Addition of two vectors is quite different from simple algebraic sum of two numbers.

- **Triangle Law of Addition of Two Vectors**

If two vectors are 'represented by two sides of a triangle in same order then their sum or resultant vector is given by the third side of the triangle taken in opposite order of the first two vectors.



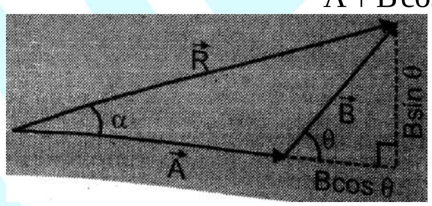
Shift one vector $\vec{B}$, without changing its direction, such that its tail coincide with head of the other vector $\vec{A}$. Now complete the triangle by drawing third side, directed from tail of $\vec{A}$ to head of $\vec{B}$ (it is in opposite order of $\vec{A}$ and $\vec{B}$ vectors).

Sum of two vectors is also called resultant vector of these two vectors. Resultant $\vec{R} = \vec{A} + \vec{B}$

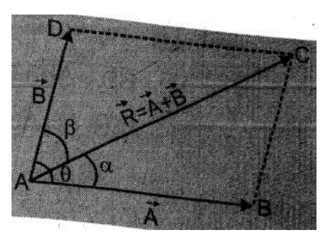
Length of $\vec{R}$ is the magnitude of vector sum i.e. $|\vec{A} + \vec{B}|$

$$\therefore \ |\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{(A + B\cos\theta)^2 + (B\sin\theta)^2} = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Let direction of $\vec{R}$ make angle α with $\vec{A}$: $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$



- **Parallelogram Law of Addition of Two Vectors :**



If two vectors are represented by two adjacent sides of a parallelogram which are directed away from their common point then their sum (i.e. resultant vector) is given by the diagonal of the parallelogram passing away through that common point.

$$\vec{AB} + \vec{AD} = \vec{AC} \Rightarrow \vec{A} + \vec{B} = \vec{R}$$

$$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}, \ \tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} \text{ and } \tan\beta = \frac{A\sin\theta}{B + A\cos\theta}$$

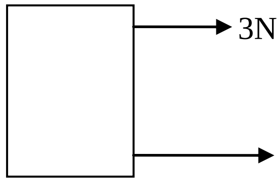
Illustration 26.

Two forces of magnitudes 3N and 4N respectively are acting on a body. Calculate the resultant force if the angle between them is :

(i) 0° (ii) 180° (iii) 90°

Solution:

(i)

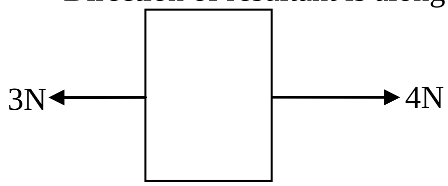


$\theta =$ 0º, both the forces are parallel, R = A + B

$\therefore$ Net force or resultant force R = 3 + 4 = 7N

Direction of resultant is along both the forces

(ii)

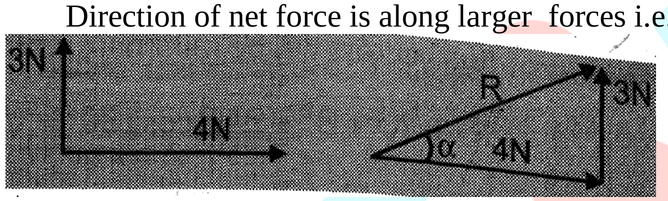


$\theta =$ 180º, both the forces are antiparallel, R = A ~ B

$\therefore$ Net force or resultant force R = 4 – 3 = 1N

Direction of net force is along larger forces i.e. along 4N.

(iii)



θ = 90º, both the forces are perpendicular

Then $R = \sqrt{A^2 + B^2 + 2AB\cos 90^o} = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5N$;

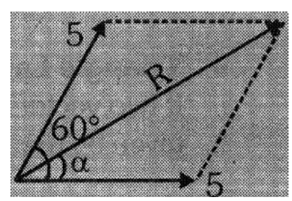
$\tan\alpha = \frac{3}{4} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 37^o$

Magnitude of resultant is 5N which is acting at an angle of 37° from 4N force.

Illustration 27.

Two vectors having equal magnitude of 5 units, have an angle of 60º between them. Find the magnitude of their resultant vector and its angle from one of the vectors.

Solution:



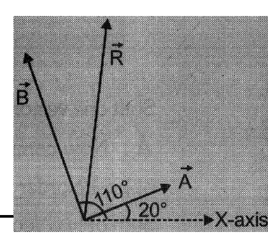
A = B = 5 unit and θ = 60º; $R = \sqrt{A^2 + B^2 + 2AB\cos 60^o} = 5\sqrt{3}$ unit

$$\tan\alpha = \frac{B\sin 60^o}{A + B\cos 60^o} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{1}{\sqrt{3}} = \tan 30^o \quad \therefore \alpha = 30^o$$

Illustration 28.

A vector $\vec{A}$ and $\vec{B}$ make angles of 20º and 110º respectively with the X-axis. The magnitudes of these vectors are 5m and 12m respectively. Find their resultant vector.

Solution:



Angle between the $\vec{A}$ and $\vec{B}$ = 110º – 20º = 90º

So, $R = \sqrt{A^2 + B^2 + 2AB\cos 90^o} = \sqrt{5^2 + 12^2}$ = 13m

Let angle of $\vec{R}$ from $\vec{A}$ is α

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta} = \frac{12\sin 90^\circ}{5+12\cos 90^\circ} = \frac{12\times 1}{5+12\times 0} = \frac{12}{5}$$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{12}{5}\right)$ with vector $\vec{A}$ or (α + 20°) with X-axis

Illustration 29.

Figure shows a parallelogram ABCD. Prove that $\vec{AC}+\vec{BD}=2\vec{BC}$

Solution:

$\vec{AC}=\vec{AB}+\vec{BC}$ & $\vec{BD}=\vec{BC}+\vec{CD}$ [applying triangle law of vectors]

Now $\vec{AC}+\vec{BD}=\vec{AB}+\vec{BC}+\vec{BC}+\vec{CD}=\vec{AB}+2\vec{BC}+\vec{CD}$

But $\vec{CD}=-\vec{AB}$ $\quad\therefore\ \vec{AC}+\vec{BD}=\vec{AB}+2\vec{BC}-\vec{AB}=2\vec{BC}$

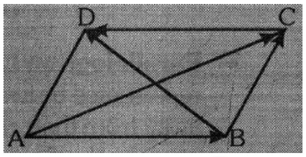
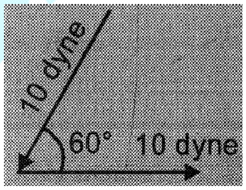


Illustration 30.

Two forces each numerically equal to 10 dynes are acting as shown in the figure, then find resultant of these two vectors.

Solution:



The angle θ between the two vectors is 120° and not 60°

$R = \sqrt{(10)^2+(10)^2+2(10)(10)(\cos 120^\circ)} = \sqrt{100+100-100} = 10$ dyne

GOLDEN KEY POINTS

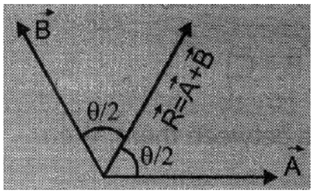
- Vector addition is commutative $\quad \vec{A}+\vec{B}=\vec{B}+\vec{A}$
- Vector addition is associative $\quad \vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$
- Resultant of two vectors will be maximum when they are parallel i.e. angle between them is zero.

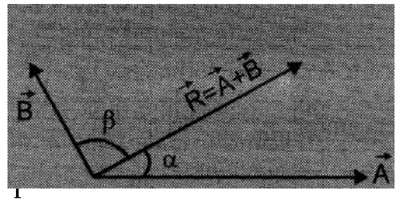
  $R_{max} = \left|\vec{A}+\vec{B}\right|_{max} = \sqrt{A^2+B^2+2AB\cos 180^\circ} = \sqrt{(A+B)^2} = A + B$ or $\left|\vec{A}+\vec{B}\right|_{max} = \left|\vec{A}\right|+\left|\vec{B}\right|$
- Resultant of two vectors will be minimum when they are parallel i.e. angle between them is 180°.

  $R_{max} = \left|\vec{A}+\vec{B}\right| = \sqrt{A^2+B^2+2AB\cos 180^\circ} = \sqrt{(A-B)^2} = A \sim B$ (larger – smaller)

  $\Rightarrow R_{min} = \left|\vec{A}\right| : \left|\vec{B}\right| \Rightarrow \left|\vec{A}+\vec{B}\right|_{min} = \left|\vec{A}\right| : \left|\vec{B}\right|$
- Resultant of two vectors of unequal magnitude can never be zero.
- If vectors are of unequal magnitude then minimum three coplanar vectors are required for zero resultant.
- Resultant of two vectors of equal magnitude will be at their bisector.

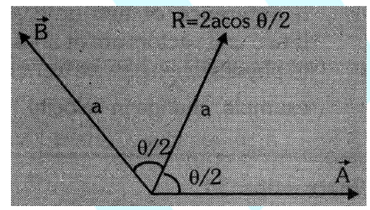
  If $\left|\vec{A}\right|=\left|\vec{B}\right|$ $\qquad$ But if $\left|\vec{A}\right|>\left|\vec{B}\right|$ then angle $\beta > \alpha$





$\vec{R}$ will incline more toward the vector of bigger magnitude.

- If two vectors have equal magnitude i.e. $|\vec{A}| = |\vec{B}|$ = a and angle between them is θ then resultant will resultant will be along the bisector $\vec{A}$ and $\vec{B}$ and its magnitude is equal 2a cos $\frac{\theta}{2}$ $|\vec{R}| = |\vec{A}+\vec{B}|$ = 2a cos $\frac{\theta}{2}$



Special Case : If θ = 0 = 120º then R = 2a cos $\frac{120^\circ}{2}$ = a

i.e. if θ = 120º then $|\vec{R}| = |\vec{A}+\vec{B}|$ = $|\vec{A}| = |\vec{B}|$ = a

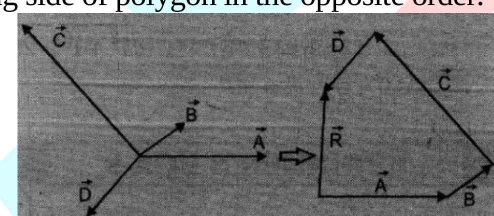
If resultant of two unit vectors is another unit vector then the angle between them (θ) = 120º

OR

If the angle between two unit vectors (θ) = 120º, then their resultant is another unit vector.

1.3 Addition of More Than Two Vectors (Law of Polygon)

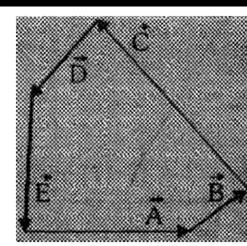
If some vectors are represented by sides of a polygon in same order, then their resultant vectors is represented by the closing side of polygon in the opposite order. $\vec{R} = \vec{A}+\vec{B}+\vec{C}+\vec{D}$



GOLDEN KEY POINTS

- In a polygon if all the vectors taken in same order are such that the head of the last vector coincides with the tail of the first vector then their resultant is a null vector.

$$\vec{A}+\vec{B}+\vec{C}+\vec{D}+\vec{E} = \vec{0}$$

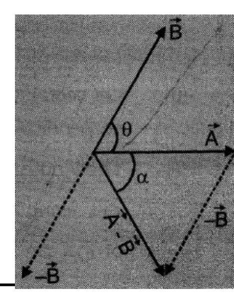


- If n coplanar vectors of equal magnitude are arranged at equal angles of separation then their resultant is always zero.

1.4 Subtraction of two vectors

Let $\vec{A}$ and $\vec{B}$ are two vectors. Their difference i.e. $\vec{A}-\vec{B}$ can be treated as sum of the vector $\vec{A}$ and vector $(-\vec{B})$.

$$\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$$



To subtract $\vec{B}$ from $\vec{A}$, reverse the direction of $\vec{B}$ and add to vector $\vec{A}$ according to law of triangle.

$|\vec{A}-\vec{B}| = \sqrt{A^2+B^2+2AB\cos(\pi-\theta)} = \sqrt{A^2+B^2-2AB\cos\theta}$ & $\tan\alpha = \dfrac{B\sin\theta}{A-B\cos\theta}$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

- Vector subtraction does not follow commutative law i.e. $\vec{A}-\vec{B} \neq \vec{B}-\vec{A}$
- Vector subtraction does not follow associative law i.e. $(\vec{A}-\vec{B})-\vec{C} \neq \vec{A}-(\vec{B}-\vec{C})$
- If two vectors have equal magnitude, i.e. $|\vec{A}| = |\vec{B}|$ = a and θ is the angle between them, then

  $|\vec{A}-\vec{B}| = \sqrt{a^2+a^2-2a^2\cos\theta}$ = $2a\sin\dfrac{\theta}{2}$

  Special case : If $\theta = 60^\circ$ then $2a\sin\dfrac{\theta}{2}$ = a i.e. $|\vec{A}-\vec{B}| = |\vec{A}| = |\vec{B}|$ = a at $\theta = 60^\circ$
- If difference of two unit vectors is another unit vector then the angle between them is 60° or If two unit vectors are at angle of 60°, then their difference is also a unit vector.
- In physics whenever we want to calculate change in a vector quantity, we have to use vector subtraction. For example, change in velocity, $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$

Illustration 31.

The magnitude of pairs of displacement vectors are given. Which pairs of displacement vectors cannot be added to give a resultant vector of magnitude 13 cm?

(1) 4 cm, 16 cm (2) 20 cm, 7 cm (3)1 cm, 15 cm (4) 6 cm, 8 cm

Ans. (3)

Solution:

Resultant of two vectors $\vec{A}$ and $\vec{B}$ must satisfy $A \sim B \leq R \leq A + B$

Illustration 32.

If $|\hat{a}+\hat{b}| = |\hat{a}-\hat{b}|$, then find the angle between $\hat{a}$ and $\hat{b}$

Solution:

Let angle between $\hat{a}$ and $\hat{b}$ be θ $\because |\hat{a}+\hat{b}| = |\hat{a}-\hat{b}|$

$\therefore \sqrt{1^2+1^2+2(1)(1)\cos\theta} = \sqrt{1^2+1^2-2(1)(1)\cos\theta}$

$\Rightarrow 2 + 2\cos\theta = 2 - 2\cos\theta \quad \Rightarrow \cos\theta = 0 \quad \because \theta = 90^\circ$

BEGINNER'S BOX - 9

1. If two forces act in opposite direction then their resultant is 10N and if they act mutually perpendicular then their resultant is 50N. Find the magnitudes of both the forces.

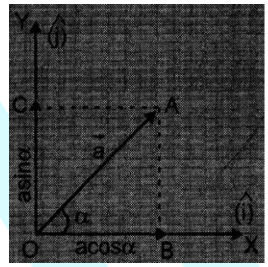
2. If $|\hat{a}-\hat{b}| = \sqrt{2}$ then calculate the value of $|\hat{a}+\sqrt{3}\hat{b}|$.

3. If $\vec{A} = 3\hat{i}+2\hat{j}$ and $\vec{B} = 2\hat{i}+3\hat{j}-\hat{k}$, then find a unit vector along $(\vec{A}-\vec{B})$.

4. If magnitude of sum of two unit vectors is $\sqrt{2}$ then find the magnitude of subtraction of these unit vectors.

1.5 Resolution of vectors into rectangular components

When a vector is splitted into components which are at right angles to each other then the components are called rectangular or orthogonal components of that vector.



(i) Let vector $\vec{a} = \overrightarrow{OA}$ in X – Y plane, makes angle α from X-axis. Draw perpendiculars AB and AC from A on the X-axis and Y-axis respectively.

(ii) The length OB is called projection of $\overrightarrow{OA}$ on X-axis or component of $\overrightarrow{OA}$ long X-axis and is represented by a_x. Similarly OC is the projection of $\overrightarrow{OA}$ on Y-axis and is represented by a_y.

According to law of vector addition. $\vec{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$

Thus $\vec{a}$ has been resolved into two parts, one along OX and the other along OY, which are mutually perpendicular.

In ΔOAB, $\frac{OB}{OA} = \cos\alpha \Rightarrow OB = OA\cos\alpha \Rightarrow a_x = a\cos\alpha$

and $\frac{AB}{OA} = \sin\alpha \Rightarrow AB = OA\sin\alpha = OC \Rightarrow a_y = a\sin\alpha$ ∴ $a_x = a\cos\alpha$ and $a_y = a\sin\alpha$

If $\hat{i}$ and $\hat{j}$ denote unit vectors along OX and OY respectively then $\overrightarrow{OB} = a\cos\alpha\,\hat{i}$ and $\overrightarrow{OC} = a\sin\alpha\,\hat{j}$ So according to rule of vector addition

$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC} \Rightarrow \vec{a} = a_x\hat{i} + a_y\hat{j} = a\cos\alpha\,\hat{i} + a\sin\alpha\,\hat{j}$

So $|\vec{a}| = \sqrt{a_x^2 + a_y^2}$ and $\tan\alpha = \frac{a_y}{a_x}$

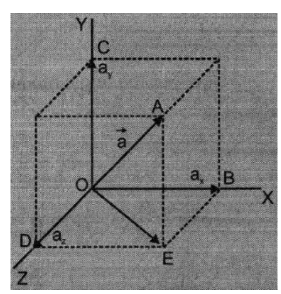
1.6 Rectangular Components of a Vector in Three Dimensions

(i) Consider a vector $\vec{a}$ represented by $\overrightarrow{OA}$, as shown in figure. Consider O as origin and draw a rectangular parallelopiped with its three edges along the X, Y and Z axes.

(ii) Vector $\vec{a}$ is the diagonal of the parallelopiped; its projections on X, Y and Z axis are $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ respectively. These are the three rectangular components of $\vec{A}$.

Using triangle law of vector addition $\overrightarrow{OA} = \overrightarrow{OE} + \overrightarrow{EA}$

Using parallelogram low of vector addition $\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{OD}$



$\therefore \overrightarrow{OA} = \left(\overrightarrow{OB} + \overrightarrow{OD}\right) + \overrightarrow{EA}$

$\because \overrightarrow{EA} = \overrightarrow{OC} \quad \therefore \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OD} + \overrightarrow{OC}$

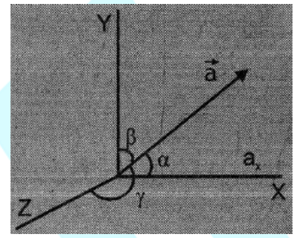
Now $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = a_x\hat{i}$, $\overrightarrow{OC} = a_y\hat{j}$ and $\overrightarrow{OD} = a_z\hat{k}$

$\therefore \vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

Also $(OA)^2 = (OE)^2 + (EA)^2$
But $(OE)^2 = (OB)^2 + (OD)^2$ and EA = OC
$\therefore (OA)^2 = (OB)^2 + (OD)^2 + (OC)^2$ or $a^2 = a_x^2 + a_y^2 + a_z^2 \Rightarrow a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ ….(i)

Direction Cosines

Let $\vec{a}$ makes angle : α with X axis, β with Y axis and γ with Z axis



- $\cos\alpha = \dfrac{a_x}{a} \Rightarrow a_x = a\cos\alpha$
- $\cos\beta = \dfrac{a_y}{a} \Rightarrow a_y = a\cos\beta$
- $\cos\gamma = \dfrac{a_z}{a} \Rightarrow a_z = a\cos\gamma$

cos α, cos β and cos γ are **direction cosines** of the vector.

Putting the value of a_x, a_y and a_z in eq. (i) we get $a^2 = a^2\cos^2\alpha + a^2\cos^2\beta + a^2\cos^2\gamma$

$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ $\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\Rightarrow 3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1$ $\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

GOLDEN KEY POINTS

- A vector can be resolved into infinite number of components.

  For example $10\hat{i} = \hat{i} + \hat{i} + \hat{i}$ ……….10 times ; = $\dfrac{\hat{i}}{2} + \dfrac{\hat{i}}{2} + \dfrac{\hat{i}}{2}$ ………..20 times and so on.
- Maximum number of rectangular components of a vector in a plane is two. But maximum number of rectangular components in space (3-dimensions) is three which are along X, Y and Z axes.
- A vector is independent of the orientation of axes but the components of that vector depend upon the orientation of axes.
- The component of a vector along its perpendicular direction is always zero.

Illustration 33.

If $\vec{P} = 3\hat{i} + 4\hat{j} + 12\hat{k}$ then find magnitude and the direction cosines of $\vec{P}$

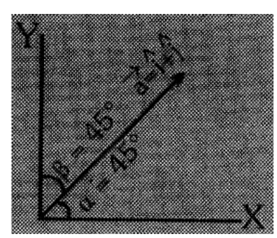
Solution:

Magnitude of $\vec{P} : |\vec{P}| = \sqrt{P_x^2 + P_y^2 + P_z^2} = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$

Direction cosines : $\cos\alpha = \dfrac{P_x}{P} = \dfrac{3}{13}$, $\cos\beta = \dfrac{P_y}{P} = \dfrac{4}{13}$, $\cos\gamma = \dfrac{P_z}{P} = \dfrac{12}{13}$

Illustration 34.

Find the angle made by $(\hat{i} + \hat{j})$ vector from X and Y axes respectively.



Solution:

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\cos\alpha = \dfrac{a_x}{a} = \dfrac{1}{\sqrt{2}} \Rightarrow \alpha = 45^\circ$ & $\cos\beta = \dfrac{a_y}{a} = \dfrac{1}{\sqrt{2}} \Rightarrow \beta = 45^\circ$

Illustration 35.

Find out the angle made by $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ vector from X, Y and Z axes respectively.

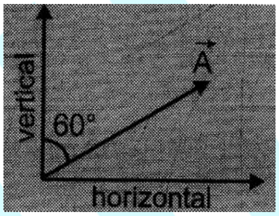
Solution:

Given $A_x = A_y = A_z = 1$ so $A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1+1+1} = \sqrt{3}$

$\cos\alpha = \frac{A_x}{A} = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\frac{1}{\sqrt{3}}$; $\cos\beta = \frac{A_y}{A} = \frac{1}{\sqrt{3}} \Rightarrow \beta = \cos^{-1}\frac{1}{\sqrt{3}}$; $\cos\gamma = \frac{A_z}{A} = \frac{1}{\sqrt{3}} \Rightarrow \cos^{-1}\frac{1}{\sqrt{3}}$

Illustration 36.

A force of 4N is inclined at an angle of 60º from the vertical. Find out its components along horizontal and vertical directions.



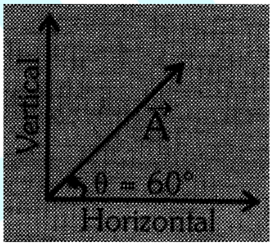
Solution:

Vertical component = 4 cos 60º = 2N

Horizontal component = 4 sin 60º = $2\sqrt{3}$ N

Illustration 37.

A force is inclined at an angle of 60º from the horizontal. If the horizontal component of the force is 40N, calculate the vertical component.



Solution:

Θ $A_x = A\cos\theta \therefore 40 = A\cos 60^\circ = \frac{A}{2} \Rightarrow A = 80N$

Now $A_y = A\sin 60^\circ = \frac{A\sqrt{3}}{2} = \frac{80\sqrt{3}}{2} = 40\sqrt{3}N$

Illustration 38.

Determine that vector which when added to the resultant of $\vec{P} = 2\hat{i} + 7\hat{j} - 10\hat{k}$ and $\vec{Q} = \hat{i} + 2\hat{j} - 3\hat{k}$ give a unit vector along X-axis.

Solution:

Resultant $\vec{R} = \vec{P} + \vec{Q} = (2\hat{i} + 7\hat{j} - 10\hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k})$ = $3\hat{i} + 9\hat{j} - 7\hat{k}$

But $\vec{R}$ + required vector = $\hat{i}$ so required vector = $\hat{i} - \vec{R} = \hat{i} - (3\hat{i} + 9\hat{j} - 7\hat{k})$ = $-2\hat{i} - 9\hat{j} + 7\hat{k}$

Illustration 39.

Add victors $\vec{A}$, $\vec{B}$ and $\vec{C}$ which have equal magnitude of 50 units and are inclined at angles of 45°, 135° and 315° respectively from X-axis.

Solution:

Angle between $\vec{B}$ and $\vec{C}$ is equal to 315° – 135° = 180°

$\therefore$ They balance each other

so sum of these three is $\vec{A}$ i.e. 50 units at 45° from X-axis

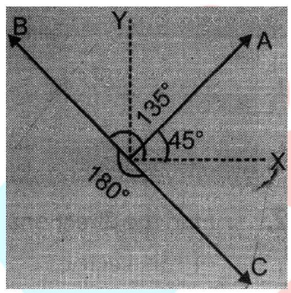
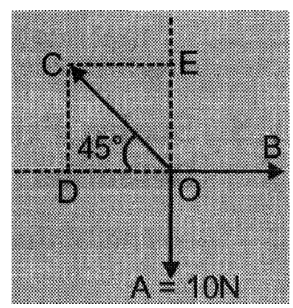


Illustration 40.

The sum of three vectors shown in figure, is zero.

What is the magnitude of vector $\vec{OB}$ & $\vec{OC}$?



Solution:

Resolve $\vec{OC}$ into two rectangular components.

OD = OC cos 45° and OE = OC sin 45°

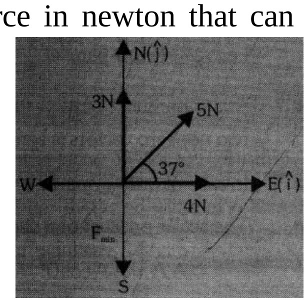
For zero resultant OE = OA or OC sin 45° = 10N

$\Rightarrow OC \times \frac{1}{\sqrt{2}} = 10N \Rightarrow |\vec{OC}| = 10\sqrt{2}N$

and OD = OB $\Rightarrow$ OC cos 45° = OB $\Rightarrow 10\sqrt{2} \times \frac{1}{\sqrt{2}}$ = OB $\Rightarrow$ OB = 10 N

Illustration 41.

For shown situation, what will be the magnitude of minimum force in newton that can be applied in any direction so that the resultant force is along east direction?



Solution:

Let force be F so resultant is in east direction

$4\hat{i} + 3\hat{j} + (5\cos 37^\circ\,\hat{i} + 5\sin 37^\circ\,\hat{j}) + \vec{F} = k\hat{i}$

$\Rightarrow 4\hat{i} + 3\hat{j} + 4\hat{i} + 3\hat{j} + \vec{F} = k\hat{i} \Rightarrow 8\hat{i} + 6\hat{j} + \vec{F} = k\hat{i}$

$\Rightarrow \vec{F} = (k - 8\hat{i}) - 6\hat{j} \Rightarrow F = \sqrt{(k-8)^2 + (6)^2} \Rightarrow F_{min} = 6N$

OR

F_{min} = y-components of existing forces

$= 3N + 5N \sin 37^o = 3N + 5N\left(\frac{3}{5}\right) = 6N$

BEGINNER'S BOX - 10

1. The x and y components of vector $\vec{A}$ are 4m and 6m respectively. The x and y components of vector $\vec{A} + \vec{B}$ are l0 m and 9m respectively. For the vector $\vec{B}$ calculate the following-
(a) x and y components (b) length and (c) the angle it makes with x-axis

2. Find the directional cosines of vector $(5\hat{i} + 2\hat{j} + 6\hat{k})$. Also write the value of sum of squares of directional cosines of this vector.

3. If $\vec{A} = 6\hat{i} - 6\hat{j} + 5\hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} - \hat{k}$, then find a unit vector parallel to the resultant of $\vec{A}$ & $\vec{B}$.

1.7 Multiplication and Division of a Vector by a Scalar

If there is a vector $\vec{A}$ and a scalar K and if $\vec{B} = K\vec{A}$ and $\vec{C} = \frac{\vec{A}}{K}$ where K > 0 then

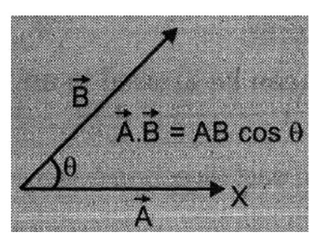
(a) In multiplication of a vector by a scalar the magnitude becomes K times while the direction remains same. So that angle between $\vec{A}$ and $\vec{B}$ is zero.
(b) In division of a vector by a scalar, the magnitude becomes (1/K) times and the direction remains same. So that angle between $\vec{A}$ and $\vec{C}$ is zero.

1.8 Scalar Product of Two Vectors

Definition: The scalar product (or dot product) of two vectors is defined as the product of their magnitudes with cosine of the angle between them. Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having angle θ between them then their scalar product is written as $\vec{A} \cdot \vec{B} = AB\cos\theta$
Example: $W = \vec{F} \cdot \vec{S}$ Where $\vec{F}$ = force and $\vec{S}$ = displacement.

GOLDEN KEY POINTS

- Dot product is always a scalar, which is positive if angle between the vectors is acute (i.e. $\theta < 90$) and negative if angle between them is obtuse (i.e. $90° < \theta \le 180$).
- Dot product is commutative $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Dot product is distributive $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- According to definition $\vec{A} \cdot \vec{B} = AB\cos\theta$ the angle between the vectors $\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{AB}\right)$



- Scalar product of two vectors will be maximum when cosθ = max = 1, i.e. θ = 0º, i.e., vectors are parallel. $(\vec{A}\cdot\vec{B})_{max} = AB$

- Scalar product of two vectors will be zero when $\cos\theta = 0$, i.e. $\theta = 90°$ so $(\vec{A}\cdot\vec{B}) = 0$ if the scalar product of two nonzero vectors is zero then vectors are orthogonal or perpendicular to each other.
- In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$: $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 1 \times 1 \times \cos 90^o = 0$
- The scalar product of a vector by itself is termed as self dot product and is given by
$$\vec{A}.\vec{A} = A\,A\cos0^o = A^2 \Rightarrow |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$
- In case of unit vector $\hat{n}$: $\hat{n}.\hat{n} = 1 \times 1 \times \cos0^o = 1$ so $\hat{n}.\hat{n} = \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$
- In terms of components: $\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}).(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) = (A_xB_x + A_yB_y + A_zB_z)$

1.9 Projection of $\vec{A}$ on $\vec{B}$

(i) In scalar form : Project of $\vec{A}$ on $\vec{B}$ $= A\cos\theta = A\left(\dfrac{\vec{A}.\vec{B}}{AB}\right) = \dfrac{\vec{A}.\vec{B}}{B} = \vec{A}.\hat{B}$

(ii) In vector form : Projection of $\vec{A}$ on $\vec{B}$ $= (A\cos\theta\ \hat{B}) = \left(\dfrac{\vec{A}.\vec{B}}{B}\right)\hat{B} = (\vec{A}.\hat{B})$

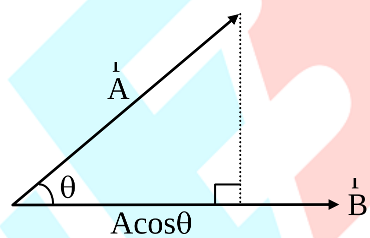


Illustration 42.

Can scalar product be ever negative?

Solution:

Yes. Scalar product will be negative if $\theta > 90^o$.

Θ $\vec{P}\cdot\vec{Q} = PQ\cos\theta$ $\therefore$ When $\theta > 90^o$ then $\cos\theta$ is negative and $\vec{P}\cdot\vec{Q}$ will be negative.

Illustration 43.

If $\vec{A} = 4\hat{i} + n\hat{j} - 2\hat{k}$ and $\vec{B} = 2\hat{i} + 3\hat{j} + \hat{k}$, then find the value of n so that $\vec{A} \perp \vec{B}$

Solution:

Dot product of two mutually perpendicular vectors is zero $\vec{A}\cdot\vec{B} = 0$

$\therefore (4\hat{i} + n\hat{j} - 2\hat{k}).(2\hat{i} + 3\hat{j} + \hat{k}) = 0 \Rightarrow (4 \times 2) + (n \times 3) + (-2 \times 1) = 0 \Rightarrow 3n = -6 \Rightarrow n = -2$

Illustration 44.

Three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are such that $\vec{A} = \vec{B} + \vec{C}$ and their magnitudes are in ratio 5 : 4 : 3 respectively. Find angle between vector $\vec{A}$ and $\vec{C}$ **[AIPMT (Mains)-2008]**

Solution:

Given that : $\vec{A} = \vec{B} + \vec{C} \Rightarrow \vec{A} - \vec{B} = \vec{C}$

By taking self dot product on both sides $(\vec{A}-\vec{C})\cdot(\vec{A}-\vec{C})=\vec{B}.\vec{B} \Rightarrow A^2 + C^2 - 2\vec{A}.\vec{C} = B^2$

Now let angle between $\vec{A}$ and $\vec{C}$ be θ then $A^2 + C^2 - 2AC\cos\theta = B^2$

$$\therefore \cos\theta = \frac{A^2+C^2-B^2}{2AC} = \frac{(5)^2+(3)^2-(4)^2}{2(5)(3)} = \frac{18}{30} = \frac{3}{5} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{5}\right) = 53^\circ$$

OR

Since $5^2 = 4^2 + 3^2$ the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ with $\vec{A} = \vec{B}+\vec{C}$ make a triangle with angle between $\vec{B}$ and $\vec{C}$ as 90°. If θ is the angle between $\vec{A}$ and $\vec{C}$, then $\cos\theta = \frac{3}{5}$ $\therefore \theta = 53^\circ$

BEGINNER'S BOX - 11

1. If $\vec{a}$ and $\vec{b}$ are two non collinear unit vectors and $|\vec{a}+\vec{b}| = \sqrt{3}$, then find the value of $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$

2. If $\vec{A} = 4\hat{i}-2\hat{j}+4\hat{k}$ and $\vec{B} = -4\hat{i}+2\hat{j}+\alpha\hat{k}$ are perpendicular to each other then find value of α?

3. If vector $(\hat{a}+2\hat{b})$ is perpendicular to vector $(5\hat{a}-4\hat{b})$, then find the angle between $\vec{a}$ and $\vec{b}$.

4. If $\vec{A}=2\hat{i}-2\hat{j}-\hat{k}$ and $\vec{B}=\hat{i}+\hat{j}$, then :

(a) Find angle between $\vec{A}$ and $\vec{B}$.

(b) Find the projection of resultant vector of $\vec{A}$ and $\vec{B}$ on x-axis.

5. Find the vector components of $\vec{a} = 2\hat{i}+3\hat{j}$ along the directions of $\hat{i}+\hat{j}$.

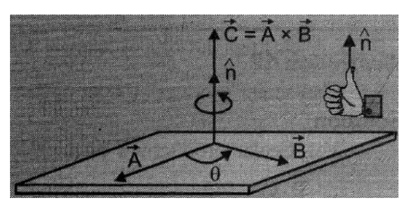
1.10 Vector Product of Two Vectors

Definition

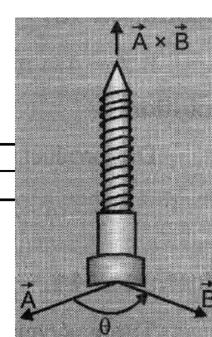
The vector product or cross product of two vectors is defined as a vector having magnitude equal to the product of their magnitudes with the sine of angle between them, and its direction is perpendicular to the plane containing both the vectors according to right hand screw rule or right hand thumb rule. If $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product i.e. $\vec{A}\times\vec{B}$ is a vector $\vec{C}$ defined by $\vec{C} = \vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$

Right Hand Thumb Rule

Place the vector $\vec{A}$ and $\vec{B}$ to tail. Now place stretched fingers and thumb of right hand perpendicular to the plane of $\vec{A}$ and $\vec{B}$ such that the fingers are along the vector $\vec{A}$. If the fingers are now closed through smaller angle so as to go towards $\vec{B}$, then the thumb gives the direction of $\vec{A}\times\vec{B}$ i.e. $\vec{C}$



Right Hand Screw Rule



The direction of $\vec{A}\times\vec{B}$ i.e. $\vec{C}$ is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and towards the advancement of a right handed screw rotated from $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus, if a right handed screw whose axis is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction $\vec{A}\times\vec{B}$.

Examples of Vector Product

(i) Torque : $\vec{\tau}=\vec{r}\times\vec{F}$

(ii) Angular momentum : $\vec{j}=\vec{r}\times\vec{p}$

(iii) Velocity : $\vec{v}=\vec{\omega}\times\vec{r}$ -

(iv) Acceleration : $\vec{a}=\vec{\alpha}\times\vec{r}$

Here r is position vector and $\vec{F},\vec{p},\vec{\omega}$ and α are force, linear momentum, angular velocity and angular acceleration respectively.

1.11 Geometrical Meaning of Vector Product of Two Vectors

(i) Consider two vectors $\vec{A}$ and $\vec{B}$ which are represented by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ and $\angle POQ = \theta$

(ii) Complete the parallelogram OPRQ. Join P with Q. Here OP=A and OQ =B. Draw QN $\perp$ OP.

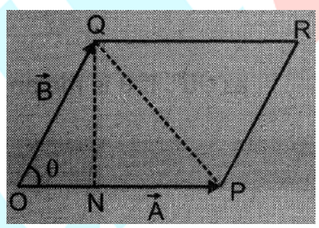
(iii) Magnitude of cross product of $\vec{A}$ and $\vec{B}$

$|\vec{A}\times\vec{B}|$ = AB sinθ = (OP) (OQ sinθ) = (OP) (NQ) ($\because$ NQ = OQ sinθ)

= base × height = Area of parallelogram OPRQ

$$\text{Area } \Delta OPQ == \frac{\text{base}\times\text{height}}{2} = \frac{(OP)(NQ)}{2} = \frac{1}{2}|\vec{A}\times\vec{B}|$$

$\therefore$ Area of parallelogram OPRQ = 2[area of Δ OPQ] = $|\vec{A}\times\vec{B}|$



Formulae to Find Area

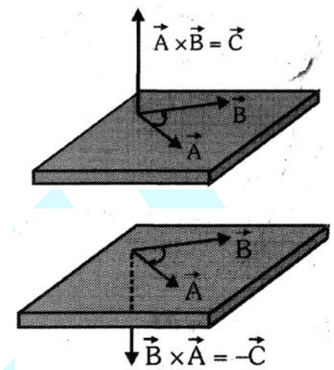
If $\vec{A}$ and $\vec{B}$ are two adjacent sides of a triangle, then its area = $\frac{1}{2}|\vec{A}\times\vec{B}|$

If $\vec{A}$ and $\vec{B}$ are two adjacent sides of a parallelogram, then its area = $|\vec{A}\times\vec{B}|$

If $\vec{A}$ and $\vec{B}$ are diagonals of a parallelogram, then its area = $\frac{1}{2}|\vec{A}\times\vec{B}|$

GOLDEN KEY POINTS

- Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors, i.e., orthogonal (perpendicular) to both the vectors $\vec{A}$ and $\vec{B}$



- Vector product of two vectors is not commutative i.e.

  $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$ But $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}|$ = AB sinθ

  Note : $\vec{A}\times\vec{B}\neq-\vec{B}\times\vec{A}$

  i.e. in case of vectors $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ magnitudes are equal but directions are opposite [See the figure]

- The vector product is distributive when the order of the vectors is strictly maintained, i.e. $\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$
- According to definition of vector product of two vectors $\vec{A}\times\vec{B}$ = AB sinθ $\hat{n}$ ⇒ θ = $\sin^{-1}\dfrac{|\vec{A}\times\vec{B}|}{AB}$
- The vector product of two vectors will be maximum when sin θ = max. = 1, i.e., θ = 90°

  $$|\vec{A}\times\vec{B}|_{max} = AB\sin 90^o = AB$$

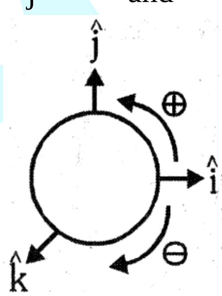
  i.e. vector product is maximum if the vectors are orthogonal (perpendicular).
- The vector product of two non-zero vectors will be zero when |sinθ| = 0.

  i.e. when θ = 0° or 180°, $|\vec{A}\times\vec{A}| = 0$ or $|\vec{A}\times(-\vec{A})| = 0$

  Therefore, if the vector product of two non-zero vectors is zero, then the vectors are collinear.
- The self cross product, i.e., product of a vector by itself is a zero vector or a null vector.

  i.e. $\vec{A}\times\vec{A}-=(AA\sin 0^o)\hat{n}=\vec{0}$
- In case of unit vector $\hat{n}$: $\hat{n}\times\hat{n}$ = 1×1×sin0° $\hat{n}$ = $\vec{0}$ so that $\hat{i}\times\hat{i}=\vec{0}$, $\hat{j}\times\hat{j}=\vec{0}$, $\hat{k}\times\hat{k}=\vec{0}$
- In case of orthogonal unit vector $\hat{i}$, $\hat{j}$ and $\hat{k}$; according to right hand thumb rule

  $\hat{i}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$ $\hat{k}\times\hat{i}=\hat{j}$ and $\hat{j}\times\hat{i}=-\hat{k}$ $\hat{k}\times\hat{j}=-\hat{i}$ $\hat{i}\times\hat{k}=-\hat{j}$



- In terms of components

  $$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\hat{i}(A_yB_z-A_zB_y)-\hat{j}(A_xB_z-A_xB_z)+\hat{k}(A_xB_y-A_yB_x)$$

- Unit vector perpendicular to $\vec{A}$ as well as $\vec{B}$ is $\hat{n}$ = $\pm\dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}=\pm\dfrac{\vec{A}\times\vec{B}}{\sin\theta}$
- If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar, than $\vec{A}\cdot(\vec{B}\times\vec{C})$ = 0. [Q $(\vec{B}\times\vec{C})$ is perpendicular to $\vec{A}$]

- Angle between $(\vec{A}+\vec{B})$ and $(\vec{A}\times\vec{B})$ is 90º as $\vec{A}\times\vec{B}$ is perpendicular to plane containing $\vec{A}$ & $\vec{B}$.
- A scalar or a vector, cannot be divided by a vector.
- Vectors of different types can be multiplied to generate new physical quantities which may be a scalar or a vector. If, in multiplication of two vectors, the generated physical quantity is a scalar, then their product is called scalar or dot product and if it is a vector, then their product is called vector or cross product.

Illustration 45.

If $\vec{F}=(4\hat{i}-10\hat{j})$ and $\vec{r}=(5\hat{i}-3\hat{j})$, then calculate torque $(\vec{\tau}=\vec{r}\times\vec{F})$.

Solution:

Here $\vec{r}=5\hat{i}-3\hat{j}+0\hat{k}$ and $\vec{F}=-4\hat{i}-10\hat{j}+0\hat{k}$

$$\therefore\ \vec{\tau}=\vec{r}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & -3 & 0\\ 4 & -10 & 0\end{vmatrix}=\hat{i}(0-0)-\hat{j}(0-0)+\hat{k}(-50+12)=-38\hat{k}$$

Illustration 46.

Find a unit vector perpendicular to both the vectors $(2\hat{i}+3\hat{j}+\hat{k})$ and $(\hat{i}-\hat{j}+2\hat{k})$.

Solution:

Let $\vec{A}=2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{B}=\hat{i}-\hat{j}+2\hat{k}$

unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ is $\hat{n}=\pm\dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 1\\ 1 & -1 & 2\end{vmatrix}=\hat{i}(6+1)-\hat{j}(4-1)+\hat{k}(-2-3)=7\hat{i}-3\hat{j}-5\hat{k}$$

$\because\ |\vec{A}\times\vec{B}|=\sqrt{7^2+(-3)^2+(-5)^2}=\sqrt{83}$ unit $\qquad \therefore\ \hat{n}=\pm\dfrac{1}{\sqrt{83}}(7\hat{i}-3\hat{j}-5\hat{k})$

Illustration 47.

If $\vec{A}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{B}=-\hat{i}+\hat{j}+4\hat{k}$ and $\vec{C}=3\hat{i}+3\hat{j}+12\hat{k}$, then find the angle between the vectors $\left(\vec{A}+\vec{B}+\vec{C}\right)$ and $\left(\vec{A}\times\vec{B}\right)$ in degrees.

Solution:

$$\text{Let } \vec{P}=\vec{A}+\vec{B}+\vec{C}=3\hat{i}-5\hat{k} \text{ and } \vec{Q}=\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -1 & 1 & 4\end{vmatrix}=5\hat{i}-7\hat{j}+3\hat{k}$$

Angle between $\vec{P}$ & $\vec{Q}$ is given by $\cos\theta=\dfrac{\vec{P}\cdot\vec{Q}}{PQ}=\dfrac{15-15}{PQ}=0\Rightarrow\theta=90^\circ$

BEGINNER'S BOX - 12

1. There are tow vectors $\vec{A} = 3\hat{i} + \hat{j}$ and $\vec{B} = \hat{j} + 2\hat{k}$. For these two vectors-

(a) If $\vec{A}$ and $\vec{B}$ are the adjacent sides of a parallelogram then find the magnitude of its area.

(b) Find a unit vector which is perpendicular to both $\vec{A}$ & $\vec{B}$.

2. If $\sqrt{3}|\vec{A}\times\vec{B}| = \vec{A}\cdot\vec{B}$, then find the angle between $\vec{A}$ and $\vec{B}$.

3. If the area of a triangle of side $\vec{A}$ & $\vec{B}$ is equal to $\frac{AB}{4}$, then find the angle between $\vec{A}$ & $\vec{B}$.

4. Find $\vec{A}\cdot\vec{B}$ if $|\vec{A}| = 2$, $|\vec{B}| = 5$, and $|\vec{A}\times\vec{B}| = 8$

ANSWER KEY

BEGINNER'S BOX – 1

1. (i) $-\frac{1}{\sqrt{3}}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{1}{\sqrt{2}}$ (iv) $-\frac{\sqrt{3}}{2}$ (v) –1 (vi) 0

2. $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$ $\cot\theta = \frac{3}{4}$ $\operatorname{cosec}\theta = \frac{5}{4}$

BEGINNER'S BOX – 2

1. 2 **2.** 13 **3.** –2/3

BEGINNER'S BOX – 3

1. (i) $\frac{7}{2}x^{5/2}$ (ii) $-3x^{-4}$ (iii) 1 (iv) $5x^4+3x^2+2x^{-1/2}$ (v) $20x^3+9x^{1/2}+9$

(vi) $2ax+b$ (vii) $15x^4-3+\frac{1}{x^2}$

2. 2t + 5 **3.** u + at **4.** 30 cm^2s^{-1} **5.** 2 πr

6. (i) 4x + 3 (ii) $-\frac{2}{(2x+1)^2}$ (iii) $-\frac{1}{(4x+5)^2}$ (iv) $\frac{2x-x^4}{(x^3+1)^2}$

BEGINNER'S BOX – 4

1. (i) $\frac{x^{16}}{16}+c$ (ii) $-2x^{-1/2}+c$ (iii) $-\frac{x^{-6}}{2}+\log_e x+c$ (iv) $\frac{x^2}{2}+2x+\log_e x+c$

(v) $\frac{x^2}{2}+\log_e x+c$ (vi) $-\frac{a}{x}+b\log_e x+c$

BEGINNER'S BOX – 5

1. (i) $\frac{GMm}{R}$ (ii) $kq_1q_2\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$ (iii) $\frac{1}{2}M(v^2-u^2)$ (iv) ∞ (vi) 1

(vii) 2

BEGINNER'S BOX – 6

1. (i) $\frac{5}{2},\frac{1}{5}$ (ii) $\frac{p}{q},\frac{q}{p}$ **2.** (4)

BEGINNER'S BOX – 7

1. 1275 **2.** 385

BEGINNER'S BOX – 8

1.	$\frac{2}{3}$	**2.**	$F_{net} = \frac{2GMm}{r^2}$

BEGINNER'S BOX – 9

1.	40N & 30N	**2.**	2	**3.**	$\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$	**4.**	$\sqrt{2}$

BEGINNER'S BOX – 10

1. (a) 6m and 3m (b) $\sqrt{45}$m (c) $\tan^{-1}\left(\frac{1}{2}\right)$

2. $\frac{5}{\sqrt{65}}, \frac{2}{\sqrt{65}}, \frac{6}{\sqrt{65}}; 1$ **3.** $\frac{7\hat{i}-4\hat{j}+4\hat{k}}{9}$

BEGINNER'S BOX – 11

1.	$\frac{1}{2}$	**2.**	5	**3.**	60^o	**4.**	(a) 90^o (b) 3	**5.**	$\frac{5}{2}(\hat{i}+\hat{j})$

BEGINNER'S BOX – 12

1. (a) 7 units (b) $\frac{2}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{3}{7}\hat{k}$ **2.** 30^o **3.** 30^o or 150^o **4.** $\vec{A}.\vec{B} = \pm 6$