# Vector

# **1.** Scalar and Vector Quantities

A physical quantity which is completely specified by its magnitude only is called **scalar**. It is represented by a real number along with suitable unit.

For example, Distance, Mass, Length, Time, Volume, Speed, Area are scalars.

On the other hand, a physical quantity which has magnitude as well as direction is called a **vector**. For example, Displacement, velocity, acceleration, force etc. are vector quantities.

## Representation of a Vector

Geometrically a vector is represented by  $\vec{a}$ directed line segment. If for a vector  $\vec{a}$ ,  $\vec{a} = \vec{AB}$ , then A is called its **initial point** and **B** is called its **terminal point**. Clearly  $\vec{AB}$  and  $\vec{BA}$ represents different line segments

If  $\vec{a} = \vec{AB}$ , then its magnitude is expressed by  $|\vec{a}|$  or  $|\vec{AB}|$  or AB.



#### Note :

The magnitude of a vector is always non negative real number.

#### 2. Kinds of Vectors

#### (i) Zero or null vector :

A vector whose magnitude is zero is called **zero** or null vector and it is denoted by 0 or  $\vec{0}$ . The initial and terminal points of the directed line segment representing zero vector are coincident and its direction is arbitrary.

#### (ii) Unit vector :

A vector of unit magnitude is called a **unit vector**. A unit vector in the direction of a is denoted by  $\hat{a}$ . Thus

$$\hat{a} = \frac{\rho}{|\hat{a}|} = \frac{\rho}{\hat{a}} = \frac{vector a}{magnitude of a}$$

Note :

- (i)  $|\hat{a}| = 1$
- (ii) Unit vectors parallel to x-axis, y-axis and z-axis are denoted by i, j and k respectively.
- (iii) Two unit vectors may not be equal unless they have the same direction.

#### (iii) Equal Vector :

Two vectors  $\vec{a}$  and  $\vec{b}$  are said to be equal, if

- (a)  $|\overrightarrow{a}| = |\overrightarrow{b}|$
- (b) they have the same direction

#### (iv) Collinear vectors or Parallel vectors :

Vectors which are parallel to the same line are called collinear vectors or parallel vectors. Such vectors have either same direction or opposite direction. If they have the same direction they are said to be like vectors, and if they have opposite directions, they are called unlike vectors.



In the diagram  $\vec{a}$  and  $\vec{c}$  are like vectors whereas

 $\vec{a}$  and  $\vec{b}$  are opposite vectors.

#### (v) Coplanar Vectors :

If the directed line segment of some given vectors lie in a plane then they are called **coplanar vectors**. It should be noted that two vectors having the same initial point are always coplanar but such three or more vectors may not be coplanar.

#### (vi) Position Vectors :

The vector OA which represents the position of the point A with respect to a fixed point (called origin) O is called position vector of the point A. If (x,y,z) are coordinates of the point A, then

$$\overrightarrow{OA} = x\hat{i} + y\hat{j} + z\hat{k}$$

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#### (vii) Reciprocal vectors :

A vector which has the same direction as vector a but whose magnitude is the reciprocal of the magnitude of a, is called the reciprocal vector of vector a and is denoted by  $a^{-1}$ .

Thus if  $a = \alpha \hat{a}$ , then

$$\mathbf{a}^{-1} = \frac{1}{\alpha} \cdot \hat{\mathbf{a}} = \frac{\alpha \hat{\mathbf{a}}}{|\vec{\mathbf{a}}|^2} = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|^2}$$

Note: A unit vector is self reciprocal.

#### 3. Addition and subtraction of Vectors

#### 3.1 Triangle law of addition :

If two vectors are represented by two consecutive sides of a triangle then their sum is represented by the third side of the triangle but in opposite direction. This is known as the **triangle law of addition of vectors**.



Thus, if 
$$\overrightarrow{AB} = \overrightarrow{a}$$
,  $\overrightarrow{BC} = \overrightarrow{b}$ , and  $\overrightarrow{AC} = \overrightarrow{c}$   
then  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$  i.e.  $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}$ 

#### 3.2 Parallelogram Law of Addition :

If two vectors are represented by two adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram whose initial point is the same as the initial point of the given vectors. This is known as **parallelogram law of addition of vectors.** 



Thus if  $\overrightarrow{OA} = \overrightarrow{a}$ ,  $\overrightarrow{OB} = \overrightarrow{b}$ , and  $\overrightarrow{OC} = \overrightarrow{c}$ 

then  $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$  i.e.  $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}$ 

Where  $\overrightarrow{OC}$  is a diagonal of the parallelogram OABC.

#### 3.3 Addition in component form :

If the vectors are defined in terms of  $\,\hat{i}\,,\,\,\hat{j}\,$  and  $\,\hat{k}\,.$ 

- i.e. if  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  and
- $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  then their sum is defined as

 $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$ 

#### Properties of Vector Addition

Vector addition has the following properties.

- (i) **Binary Operation :** The sum of two vectors is always a vector.
- (ii) Commutativity :

For any two vectors  $\vec{a}$  and  $\vec{b}$ ,

$$\vec{a} + \vec{b} = \vec{b} + \vec{c}$$

(iii) Associativity :

For any three vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ ,  $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ 

(iv) Identity :

zero vector is the identity for addition For any vector a

$$\vec{0} + \vec{a} = \vec{a} = \vec{a} + \vec{0}$$

## (v) Additive inverse :

For every vector its negative vector  $-\vec{a}$  exists. such that

 $\overrightarrow{a}$  + (- $\overrightarrow{a}$ ) = (- $\overrightarrow{a}$ ) +  $\overrightarrow{a}$  =  $\overrightarrow{0}$ 

i.e. (-a) is the additive inverse of the vector  $\vec{a}$ .

Also if  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ 

then  $-\overrightarrow{a} = a_1 \hat{i} - a_2 \hat{j} - a_3 \hat{k}$ 

(vi) Cancellation Law :

For any three vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  $\vec{a} + \vec{b} = \vec{a} + \vec{c}$  $\vec{b} + \vec{a} = \vec{c} + \vec{a}$   $\Rightarrow \vec{b} = \vec{c}$ 

# Subtraction of Vectors

If  $\vec{a}$  and  $\vec{b}$  are two vectors, then their subtraction  $\vec{a} - \vec{b}$  is defined as  $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$ where  $-\vec{b}$  is the negative of  $\vec{b}$  having magnitude equal to that of  $\vec{b}$  and direction opposite to  $\vec{b}$ .

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If 
$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$
  
and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$   
then  $\vec{a} - \vec{b} = (\vec{a}_1 - \vec{b}_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 + b_3)\hat{k}$ 

Note :



- (i)  $\vec{a} \vec{b} \neq \vec{b} \vec{a}$
- (ii)  $(\overrightarrow{a} \overrightarrow{b}) \overrightarrow{c} \neq \overrightarrow{a} (\overrightarrow{b} \overrightarrow{c})$
- (iii) Since any one side of a triangle is less than the sum and greater than the difference of the

other two sides, so for any two vectors  $\vec{a}$  and

 $\vec{b}$ , we have

$$\begin{vmatrix} \vec{a} + \vec{b} \end{vmatrix} \le \begin{vmatrix} \vec{a} \end{vmatrix} + \begin{vmatrix} \vec{b} \end{vmatrix}$$
$$\begin{vmatrix} \vec{a} + \vec{b} \end{vmatrix} \ge \begin{vmatrix} \vec{a} \end{vmatrix} - \begin{vmatrix} \vec{b} \end{vmatrix}$$
$$\begin{vmatrix} \vec{a} - \vec{b} \end{vmatrix} \le \begin{vmatrix} \vec{a} \end{vmatrix} + \begin{vmatrix} \vec{b} \end{vmatrix}$$

 $|\overrightarrow{a} - \overrightarrow{b}| \ge |\overrightarrow{a}| - |\overrightarrow{b}|$ 

# Vectors in terms of Position Vectors of end Point

If AB be any given vector and also suppose that the position vectors of initial point A and terminal point B are a and b respectively,

then  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{b} - \overrightarrow{a}$ 

i.e. AB = p.v. of point B - p.v. of point A

= p.v. of terminal point - p.v. of initial point

# **Distance between two Points**

Let A and B be two given points whose coordinate are respectively  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ 

If  $\vec{a}$  and  $\vec{b}$  are p.v. of A and B relative to point O, then

$$\overrightarrow{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

$$\vec{b} = \mathbf{x}_2 \,\hat{\mathbf{i}} + \mathbf{y}_2 \,\hat{\mathbf{j}} + \mathbf{z}_2 \,\hat{\mathbf{k}}$$

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Now 
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{b} - \overrightarrow{a}$$
  
=  $(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$ 

Distance between the points and

= magnitude of 
$$\overrightarrow{AB}$$
  
=  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ 

# Multiplication of a Vector by a Scalar

If  $\vec{a}$  is a vector and m is a scalar (i.e. a real number) then ma is a vector whose magnitude is

m times that of  $\vec{a}$  and whose direction is the same as that of a, if m is positive and opposite to that of

- $\vec{a}$ , if m is negative,
- $\therefore$  magnitude of m  $\vec{a} = |m \vec{a}|$
- $\Rightarrow$  m (magnitude of a) = m  $|\vec{a}|$
- Again if  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$  then

$$\vec{\mathbf{n} \mathbf{a}} = (\mathbf{m}\mathbf{a}_1)\hat{\mathbf{i}} + (\mathbf{m}\mathbf{a}_2)\hat{\mathbf{j}} + (\mathbf{m}\mathbf{a}_3)\hat{\mathbf{k}}$$

#### Note :

- (i) The multiplication of a vector by a scalar is also named as 'scalar multiplication'.
- (ii) From the definition of Scalar multiplication it is obvious to note that

 $\vec{a} \parallel \vec{b} \Rightarrow \vec{a} = m \vec{b}$ , where m is some suitable scalar.

#### **Properties**

If  $\vec{a}$ ,  $\vec{b}$  are any two vectors and m, n are any scalar then

(i) m(
$$\vec{a}$$
) = ( $\vec{a}$ ) m = m $\vec{a}$  (commutativity)  
(ii) m (n $\vec{a}$ ) = n (m $\vec{a}$ ) = (mn) $\vec{a}$  (Associativity)  
(iii) (m + n) $\vec{a}$  = m $\vec{a}$  + n $\vec{a}$   
(iv) m ( $\vec{a}$  +  $\vec{b}$ ) = m $\vec{a}$  + m $\vec{b}$  (Distributivity)

## Position Vector of a Dividing Point

If  $\vec{a}$  and  $\vec{b}$  are the position vectors of two points A and B, then the position vector c of a point P dividing AB in the ratio m : n is given by



#### Particular Case :

- (i) Position vector of the mid point of AB is  $\frac{b+b}{2}$
- (ii) Any vector along the internal bisector of ∠AOB is given by
  - $\lambda(\vec{a} + \vec{b})$

#### Note :

(i) If the point P divides AB in the ratio m: n externally, then m/n will be negative. If m is positive and n is negative, then p.v.c of P is

given by 
$$\vec{c} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

(ii) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are position vectors of vertices of a triangle, then p.v. of its centroid is

$$\frac{\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}}{2}$$

(iii) If a,b,c,d are position vectors of vertices of a tetrahedron, then p.v. of its centroid is

 $\frac{\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} + \overrightarrow{d}}{4}$ 

## . Relation between two parallel vectors

(i) If  $\vec{a}$  and  $\vec{b}$  be two parallel vectors, then there exists a scalar k such that  $\vec{a} = k \vec{b}$ i.e. there exist two non- zero scalar quantities x and y so that  $x \vec{a} + y \vec{b} = 0$  If a and b be two non-zero non-parallel vectors then  $\vec{x a} + \vec{b} = 0 \implies x = 0$  and y = 0obviously

$$\mathbf{x} \stackrel{\rightarrow}{\mathbf{a}} + \mathbf{y} \stackrel{\rightarrow}{\mathbf{b}} = 0 \implies \begin{cases} \stackrel{\rightarrow}{\mathbf{a}} = 0, \stackrel{\rightarrow}{\mathbf{b}} = 0 \\ \text{or} \\ \mathbf{x} = 0, \mathbf{y} = 0 \\ \text{or} \\ \stackrel{\rightarrow}{\mathbf{a}} \parallel \stackrel{\rightarrow}{\mathbf{b}} \end{cases}$$

(ii) If 
$$\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$
 and  
 $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$  then from the property

of parallel vector, we have  $\vec{a} \parallel \vec{b}$ 

$$\Rightarrow \frac{\mathbf{a}_1}{\mathbf{b}_1} = \frac{\mathbf{a}_2}{\mathbf{b}_2} = \frac{\mathbf{a}_3}{\mathbf{b}_3}$$

# 5. Collinearity of three Points

(i) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be position vectors of three points A,B and C respectively and x, y,z be three scalars so that all are not zero, then the necessary and sufficient conditions for three points to be collinear is that

 $\vec{x a} + \vec{y b} + \vec{z c} = 0$  and  $x + \vec{y} + \vec{z} = 0$ 

(ii) Three points A, B and C are collinear, if any two vectors  $\overrightarrow{AB}$ ,  $\overrightarrow{BC}$  and  $\overrightarrow{CA}$  are parallel i.e. one of them is scalar multiple of any one of the remaining vectors.

# 6. Coplanar & Non-Coplanar Vector

(i) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be three coplanar vectors, then a vector c can be expressed uniquely as linear combination of remaining two vectors i.e.

$$\vec{c} = \lambda \vec{a} + \mu \vec{b}$$

Where  $\lambda$  and  $\mu$  are suitable scalars.

Again  $\vec{c} = \lambda \vec{a} + \mu \vec{b} \Rightarrow$  vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$  are coplanar.

If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be three coplanar vectors, then there exist three non zero scalars x, y, z so that

$$\overrightarrow{x a} + \overrightarrow{b} + \overrightarrow{z c} = 0$$

(ii) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be three non coplanar non zero vector then

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$$\vec{x a} + \vec{y b} + \vec{z c} = 0 \implies x = 0, y = 0, z = 0$$

(iii) Any vector r can be expressed uniquely as the linear combination of three non coplanar and non-

zero vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  i.e.  $r = x \vec{a} + y \vec{b} + z \vec{c}$ where x, y and z are scalars.

# **Product of Vectors**

Product of two vectors is done by two methods when the product of two vectors results in a scalar quantity then it is called scalar product. It is also called as dot product because this product is represented by putting a dot.

When the product of two vectors results in a vector quantity then this product is called Vector Product. This product is represented by (x) sign so that it is also called as **cross product**.

# Scalar or dot product of two Vectors

#### 7.1 Definition :

If  $\vec{a}$  and  $\vec{b}$  are two non zero vectors and  $\theta$  be the angle between them, then their scalar product (or dot product) is defined as the number a b  $\cos \theta$ 

where a and b are modulii of  $\vec{a}$  and  $\vec{b}$ respectively and  $0 \le \theta \le \pi$ . It is denoted by  $\vec{a} \cdot \vec{b}$ .





Note :

- (i)  $\vec{a} \cdot \vec{b} \in \mathbf{R}$
- (ii)  $\vec{a} \cdot \vec{b} \leq |\vec{a}| |\vec{b}|$
- (iii)  $\vec{a} \cdot \vec{b} > 0 \Rightarrow$  angle between  $\vec{a}$  and  $\vec{b}$  is acute
  - $\vec{a} \cdot \vec{b} < 0 \Rightarrow$  angle between  $\vec{a}$  and  $\vec{b}$  is obtuse.
- (iv) The dot product of  $\vec{a}$  zero and non-zero vector is a scalar zero.

#### 7.2 Geometrical Interpretation :

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Geometrically, the scalar product of two vectors is equal to the product of the magnitude of one and the projection of second in the direction of first

vector i.e.  $\vec{a} \cdot \vec{b} = \vec{a} (\vec{b} \cos \theta)$ 

 $= |\vec{a}|$  (projection of  $\vec{b}$  in the direction of  $\vec{a}$ )

Similarly  $\vec{a} \cdot \vec{b} = \vec{b} (\vec{a} \cos \theta)$ 

 $= |\vec{b}|$  (projection of  $\vec{a}$  in the direction of  $\vec{b}$ )

Here projection of  $\vec{b}$  on  $\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{a}}$ Projection of  $\vec{a}$  on  $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ 

# Scalar product in particular cases

- (i) If  $\vec{a}$  and  $\vec{b}$  are like vectors, then  $\theta = 0$  so  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| = \vec{a} \vec{b}$  i.e. scalar product of two like vectors is equal to the product of their modulii
- (ii) If  $\vec{a}$  and  $\vec{b}$  are unlike vectors then  $\theta = \pi$  so

 $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b} \cos \pi = -\vec{a} \cdot \vec{b}$ .

(iii) The scalar product of a vector by itself is equal to the square of its modulus i.e.

 $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ 

(iv) If  $\vec{a}$  and  $\vec{b}$  are perpendicular to each other then  $\theta = \pi/2$ , so

 $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b} \cos \pi/2 = 0$ 

i.e. the scalar product of two perpendicular vectors is always zero.

But its converse may not be true i.e.

 $\vec{a} \cdot \vec{b} = 0 \implies \vec{a} \perp \vec{b}$ 

But if a and b are non zero vectors, then

$$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$$

Thus  $\vec{a} \neq 0$ ,  $\vec{b} \neq 0$ ,  $\vec{a} \cdot \vec{b} = 0$ ,  $\Rightarrow \vec{a} \perp \vec{b}$ 

- (v) With the help of the above cases, we get the following important results:
- (a)  $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$  (b)  $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} \cdot \hat{k} \cdot \hat{j} = 0$
- (vi) If  $\vec{a}$  and  $\vec{b}$  are unit vectors, then

 $\vec{a} \cdot \vec{b} = \cos \theta$ 

# **Properties of Scalar Product**

If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are any vectors and m, n any scalars then

(i)  $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$  (Commutativity) (ii)  $(m \vec{a}) \cdot \vec{b} = \vec{a} \cdot (m \vec{b}) = m (\vec{a} \cdot \vec{b})$ (iii)  $(m \vec{a}) \cdot (n \vec{b}) = (mn) (\vec{a} \cdot \vec{b})$ (iv)  $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$  (Distributivity) (v)  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \implies \vec{b} = \vec{c}$ Infact  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \implies \vec{a} \cdot (\vec{b} - \vec{c}) = 0$  $\implies \vec{a} = 0$  or  $\vec{b} = \vec{c}$  or  $\vec{a} \perp (\vec{b} - \vec{c})$ 

(vi)  $(\vec{a}, \vec{b}), \vec{c}$  is meaningless

(vi) scalar product is not binary operation.

## Note :

(a)  $(\vec{a}, \vec{b})$ .  $\vec{b}$  is not defined

(b) 
$$(\vec{a} + \vec{b})^2 = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$$
  
(c)  $(\vec{a} - \vec{b})^2 = |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$   
(d)  $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = |\vec{a}|^2 - |\vec{b}|^2$   
(e)  $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}| \Rightarrow \vec{a} ||\vec{b}$   
(f)  $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Rightarrow \vec{a} + \vec{b}$ 

(g)  $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \Rightarrow \vec{a} \perp \vec{b}$ 

# Scalar product in terms of Components

Let  $\vec{a}$  and  $\vec{b}$  be two vectors such that  $\vec{a} = \vec{a}_1 \hat{i} + \vec{a}_2 \hat{j} + \vec{a}_3 \hat{k}$ and  $\vec{b} = \vec{b}_1 \hat{i} + \vec{b}_2 \hat{j} + \vec{b}_3 \hat{k}$ Then  $\vec{a} \cdot \vec{b} = \vec{a}_1 \vec{b}_1 + \vec{a}_2 \vec{b}_2 + \vec{a}_3 \vec{b}_3$ In particular  $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = \vec{a}_1^2 + \vec{a}_2^2 + \vec{a}_3^2$ 

For any vector 
$$\vec{a}$$
,

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 $\overrightarrow{\mathbf{b}}$  |<sup>2</sup>

# Angle between two Vectors

 $\vec{a} = (\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k}$ 

(i) If  $\vec{a}$  and  $\vec{b}$  be two vectors and  $\theta$  be the angle between them, then

$$\cos \theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|} = \frac{\overrightarrow{a}}{a} \cdot \frac{\overrightarrow{b}}{b} = \hat{a} \cdot \hat{b}$$
(ii) If  $\overrightarrow{a} = \overrightarrow{a_1} \cdot \hat{i} + \overrightarrow{a_2} \cdot \hat{j} + \overrightarrow{a_3} \cdot \hat{k}$  and  
 $\overrightarrow{b} = \overrightarrow{b_1} \cdot \hat{i} + \overrightarrow{b_2} \cdot \hat{j} + \overrightarrow{b_3} \cdot \hat{k}$  then  

$$\cos \theta = \frac{\overrightarrow{a_1} \cdot \overrightarrow{b_1} + \overrightarrow{a_2} \cdot \overrightarrow{b_2} + \overrightarrow{a_3} \cdot \overrightarrow{b_3}}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Note: If  $\vec{a}$  and  $\vec{b}$  are perpendicular to each other then  $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$ 



If a constant force F acting on a particle displaces it from point A to B, then

work done by the force W = F.  $\vec{d}$ 

# (where $\vec{d} = \vec{AB}$ )

# 8. Vector or cross product of two Vectors



#### 8.1 Definition :

If  $\vec{a}$  and  $\vec{b}$  be two vectors and  $\theta (0 \le \theta \le \pi)$  be the angle between them, then their vector (or cross) product is defined to be a vector whose magnitude is ab sin  $\theta$  and whose

direction is perpendicular to the plane of  $\vec{a}$  and

 $\vec{b}$  such that  $\vec{a}$ ,  $\vec{b}$  and  $\vec{a} \times \vec{b}$  form a right handed system.

$$\therefore \vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$
$$= \vec{a} \vec{b} \sin \theta \hat{n}$$

Where  $\hat{n}$  is a unit vector perpendicular to the plane of  $\vec{a}$  and  $\vec{b}$  such that  $\vec{a}$ ,  $\vec{b}$  and  $\hat{n}$  form a right handed system.

## 8.2 Vector product in terms of components :

If 
$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
 and  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ 

then 
$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2) \hat{i} + (a_3 b_1 - a_1 b_3) \hat{j}$$

$$+ (a_{1} b_{2} - a_{2} b_{1}) \hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_{1} & a_{2} & a_{3} \\ b_{1} & b_{2} & b_{3} \end{vmatrix}$$

## 8.3 Angle between two vectors :

If  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$ , then

$$\sin \theta = \frac{|\stackrel{\rightarrow}{a} \times \stackrel{\rightarrow}{b}|}{|\stackrel{\rightarrow}{a} || \stackrel{\rightarrow}{b}|}$$

If  $\hat{n}$  is the unit vector perpendicular to the plane

of  $\vec{a}$  and  $\vec{b}$ , then

$$\hat{\mathbf{n}} = \frac{\overrightarrow{\mathbf{a} \times \mathbf{b}}}{|\overrightarrow{\mathbf{a} \times \mathbf{b}}|}$$

# Vector product in Particular Cases

(i) The vector product of two parallel vectors is always zero i.e. if vectors a and b are parallel, then  $\vec{a} \times \vec{b} = 0$ 

In particular  $\vec{a} \times \vec{b} = 0$ 

(ii) If  $\vec{a}$  and  $\vec{b}$  are perpendicular vectors, then  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \hat{n} = \vec{a} \vec{b} \hat{n}$ 

(iii) If  $\hat{i}, \hat{j}, \hat{k}$  be three mutually perpendicular unit vectors, then

- (a)  $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$
- (b)  $\hat{i} \times \hat{j} = \hat{k}$ ,  $\hat{i} \times \hat{k} = \hat{i}$ ,  $\hat{k} \times \hat{i} = \hat{i}$

(c) 
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \ \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \ \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$$

Expression for sin θ

If 
$$\vec{a} = \vec{a}_1 \hat{i} + \vec{a}_2 \hat{j} + \vec{a}_3 \hat{k}$$
  
and  $\vec{b} = \vec{b}_1 \hat{i} + \vec{b}_2 \hat{j} + \vec{b}_3 \hat{k}$ 

and  $\theta$  be angle between  $\vec{a}$  and  $\vec{b}$ , then

$$\sin^2 \theta = \frac{(a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$$

# Properties of Vector Product

- If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are any vectors and m,n any scalars then
- (i)  $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$  (Non-commutativity) but  $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ and  $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}|$ (ii)  $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$ (iii)  $(m\vec{a}) \times (n\vec{b}) = (mn)(\vec{a} \times \vec{b})$

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(iv)  $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ 

(v) 
$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$
  
(Distributivity)

(vi)  $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \implies \vec{b} = \vec{c}$  Infact

 $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \implies \vec{a} \times (\vec{b} - \vec{c}) = 0$ 

 $\Rightarrow \vec{a} = 0 \text{ or } \vec{b} = \vec{c} \text{ or } \vec{a} \parallel (\vec{b} - \vec{c})$ 

# Geometrical interpretation of Vector **Product**

The vector product of the vectors  $\vec{a}$  and  $\vec{b}$ represents a vector whose modulus is equal to the area of the parallelogram whose two adjacent sides are represented by  $\vec{a}$  and  $\vec{b}$ . Therefore  $|\vec{a} \times \vec{b}| =$ area of a parallelogram whose adjacent sides are  $\overrightarrow{a}$  and  $\overrightarrow{b}$ .

Further it should be noted that if  $\vec{a}$ ,  $\vec{b}$  represent two diagonals of a parallelogram,

then the area of the parallelogram =  $\frac{1}{2} |\vec{a} \times \vec{b}|$ 

Note : Area of a quadrilateral ABCD

 $=\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$ 

# > Area of a Triangle

(i) Area of triangle ABC =  $\frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$ 

(ii) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are position vectors of vertices of a  $\triangle$ ABC then its

Area = 
$$\frac{1}{2} |(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})|$$

Note :

Three points with position vectors  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are collinear if

 $(\overrightarrow{a} \times \overrightarrow{b}) + (\overrightarrow{b} \times \overrightarrow{c}) + (\overrightarrow{c} \times \overrightarrow{a}) = 0$ 

Moment of a Force

The moment of the force F acting at a point A about O is given by

Moment of  $F = \overrightarrow{OA} \times F = \overrightarrow{r} \times F$ 

# 9. Scalar Triple Product

# 9.1 Definition :

If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are three vectors, then their scalar triple product is defined as the dot product of two vectors  $\vec{a}$  and  $\vec{b} \times \vec{c}$ . It is generally denoted by  $\vec{a}$ .  $(\vec{b} \times \vec{c})$  or  $[\vec{a} \vec{b} \vec{c}]$ . It is read as box product of  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ . Similarly other scalar triple products be defined can as  $(\vec{b} \times \vec{c}) \cdot \vec{a} \cdot (\vec{c} \times \vec{a}) \cdot \vec{b}$ .

# Note :

Scalar triple product always results in a scalar quantity (number).

# 9.2 Geometrical Interpretation :

The scalar triple product of three vectors is equal to the volume of the parallelopiped whose three coterminous edges are represented by the given vector.

Therefore  $(\vec{a} \times \vec{b})$ .  $\vec{c} = [\vec{a} \ \vec{b} \ \vec{c}] =$ Volume of the parallelopiped whose coterminous edges are  $\overrightarrow{a}$ ,  $\overrightarrow{b}$  and  $\overrightarrow{c}$ 

# 9.3 Formula for scalar Triple Product :

(i) If  $\overrightarrow{a} = a_1\lambda + a_2m + a_3n$ ,  $\overrightarrow{b} = b_1\lambda + b_2m + b_3n$ and  $\vec{c} = c_1 \lambda + c_2 m + c_3 n$ , then

$$\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\lambda mn]$$

(ii) 
$$\vec{a} = \vec{a}_1 \hat{i} + \vec{a}_2 \hat{j} + \vec{a}_3 \hat{k}$$
,  $\vec{b} = \vec{b}_1 \hat{i} + \vec{b}_2 \hat{j} + \vec{b}_3 \hat{k}$  and  $\vec{c} = \vec{c}_1 \hat{i} + \vec{c}_2 \hat{j} + \vec{c}_3 \hat{k}$ , then  

$$\begin{vmatrix} a_1 & a_2 & a_3 \end{vmatrix}$$

$$\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(iii) For any three vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$ 

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(a)  $\begin{bmatrix} \vec{a} + \vec{b} & \vec{b} + \vec{c} & \vec{c} + \vec{a} \end{bmatrix} = 2 \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix}$ (b)  $\begin{bmatrix} \vec{a} - \vec{b} & \vec{b} - \vec{c} & \vec{c} - \vec{a} \end{bmatrix} = 0$ (c)  $\begin{bmatrix} \vec{a} \times \vec{b} & \vec{b} \times \vec{c} & \vec{c} \times \vec{a} \end{bmatrix} = 2 \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix}^2$ 

#### 9.4 Properties of Scalar Triple product

(i) The position of (.) and (×) can be interchanged

i.e. 
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$
  
but  $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{c} \cdot (\vec{a} \times \vec{b})$ 

So  $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{b} & \vec{c} & \vec{a} \end{bmatrix} = \begin{bmatrix} \vec{c} & \vec{a} & \vec{b} \end{bmatrix}$ 

Therefore if we don't change the cyclic order of a, b and c then the value of scalar triple product is not changed by interchanging dot and cross.

(ii) If the cyclic order of vectors is changed, then sign of scalar triple product is changed i.e.

$$\vec{a} \cdot [\vec{b} \times \vec{c}] = -\vec{a} \cdot (\vec{c} \times \vec{b})$$

or  $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = -\begin{bmatrix} \vec{a} & \vec{c} & \vec{b} \end{bmatrix}$ from (i) and (ii) we have

$$\begin{bmatrix} \overrightarrow{a} & \overrightarrow{b} & \overrightarrow{c} \end{bmatrix} = \begin{bmatrix} \overrightarrow{b} & \overrightarrow{c} & \overrightarrow{a} \end{bmatrix} = \begin{bmatrix} \overrightarrow{c} & \overrightarrow{a} & \overrightarrow{b} \end{bmatrix}$$

$$=-\begin{bmatrix} \vec{a} & \vec{c} & \vec{b} \end{bmatrix} = -\begin{bmatrix} \vec{b} & \vec{a} & \vec{c} \end{bmatrix} = -\begin{bmatrix} \vec{c} & \vec{b} & \vec{a} \end{bmatrix}$$

(iii) The scalar triple product of three vectors when two of them are equal or parallel, is zero i.e.

 $\begin{bmatrix} \vec{a} & \vec{b} & \vec{b} \end{bmatrix} = \begin{bmatrix} \vec{a} & \vec{b} & \vec{a} \end{bmatrix} = 0$ 

(iv) The scalar triple product of three mutually perpendicular unit vectors is ±1 Thus

 $[\hat{i} \ \hat{j} \ \hat{k}] = 1, [\hat{i} \ \hat{k} \ \hat{j}] = -1$ 

- (v) If two of the three vectors  $\vec{a}, \vec{b}, \vec{c}$  are parallel then  $[\vec{a}, \vec{b}, \vec{c}] = 0$
- (vi)  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are three coplanar vectors if
  - $[\vec{a} \ \vec{b} \ \vec{c}] = 0$  i.e. the necessary and sufficient condition for three non-zero collinear vectors to be coplanar is
    - $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = 0$

(vii) For any vectors 
$$\vec{a}$$
,  $\vec{b}$ ,  $\vec{c}$ , d

$$\begin{bmatrix} \vec{a} + \vec{b} & \vec{c} & d \end{bmatrix} = \begin{bmatrix} \vec{a} & \vec{c} & d \end{bmatrix} + \begin{bmatrix} \vec{b} & \vec{c} & d \end{bmatrix}$$

**10.** Volume of Tetrahedron

Power by: VISIONet Info Solution Pvt. Ltd Website : www.edubull.com (i) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  are position vectors of vertices A, B and C with respect to O, then volume of tetrahedron OABC

$$= \frac{1}{6} \begin{bmatrix} \overrightarrow{a} & \overrightarrow{b} & \overrightarrow{c} \end{bmatrix}$$

(ii) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$ ,  $\vec{d}$  are position vectors of vertices A,B,C,D of a tetrahedron ABCD, then

its volume = 
$$\begin{cases} \frac{1}{6} \begin{bmatrix} \overrightarrow{AB} & \overrightarrow{AC} & \overrightarrow{AD} \end{bmatrix} \\ & \text{or} \\ \frac{1}{6} \begin{bmatrix} b-a & c-a & d-a \end{bmatrix} \end{cases}$$

# **11. Vector Triple Product**

#### **11.1 Definition :**

The vector triple product of three vectors  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  is defined as the vector product of two vectors  $\vec{a}$  and  $\vec{b} \times \vec{c}$ . It is denoted by  $\vec{a} \times (\vec{b} \times \vec{c})$ .

#### **11.2 Properties :**

(i) Expansion formula for vector triple product is given by  $\hat{a} \times (\vec{b} \times \hat{c}) = (\hat{a} \cdot \hat{c}) \vec{b} - (\hat{a} \cdot \hat{b}) \hat{c}$   $(\vec{b} \times \hat{c}) \times \hat{a} = (\vec{b} \cdot \hat{a}) \hat{c} - (\hat{c} \cdot \hat{a}) \hat{b}$ . (ii) If  $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ ,  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  and  $\hat{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$  then  $\hat{a} \times (\vec{b} \times \hat{c})$  $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$ 

Note :

(i) Vector triple product is a vector quantity.

(ii)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ 

- (iii) Vector  $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$  is coplanar with  $\overrightarrow{b}$  and  $\overrightarrow{c}$
- (iv) The direction of  $\stackrel{\rho}{a} \times (\stackrel{\rho}{b} \times \stackrel{\rho}{c})$  is perpendicular
  - to  $\vec{a}$  and parallel to and  $\vec{b}$  and  $\vec{c}$ .