Quadratic Equation

1. Polynomial

Algebraic expression containing many terms is called Polynomial.

e.g $4x^4 + 3x^3 - 7x^2 + 5x + 3$, $3x^3 + x^2 - 3x + 5$

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots.. a_{n-1} x^{n-1} + a_nx^n$

where x is a variable, $a_0, a_1, a_2 \ldots\ldots\ldots. a_n \in C$.

1.1 Real Polynomial : Let a_0, $a_1, a_2 \ldots.. a_n$ be real numbers and x is a real variable.

Then $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots.. + a_n x^n$ is called real polynomial of real variable x with real coefficients.

eg. $-3x^3 - 4x^2 + 5x - 4$, $x^2 - 2x + 1$ etc. are real polynomials.

1.2 Complex Polynomial: If $a_0, a_1, a_2 \ldots a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots a_nx^n$ is called a complex polynomial of complex variable x with complex coefficients.

eg.- $3x^2 - (2+4i)x + (5i-4)$, $x^3 - 5ix^2 + (1+2i)x + 4$ etc. are complex polynomials.

1.3 Degree of Polynomial : Highest Power of variable x in a polynomial is called as a degree of polynomial.

e.g. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots. a_{n-1}x^{n-1} + a_nx^n$ is n degree polynomial.

$f(x) = 4x^3 + 3x^2 - 7x + 5$ is 3 degree polynomial

$f(x) = 3x - 4$ is single degree polynomial or Linear polynomial.

$f(x) = bx$ is odd Linear polynomial

2. Quadratic Expression

A Polynomial of degree two of the form $ax^2 + bx + c$ $(a \neq 0)$ is called a quadratic expression in x.

e.g $3x^2 + 7x + 5$, $x^2 - 7x + 3$

General form : – $f(x) = ax^2 + bx + c$

where $a, b, c \in C$ and $a \neq 0$

3. Quadratic Equation

A quadratic Polynomial f(x) when equated to zero is called Quadratic Equation.

e.g $3x^2 + 7x + 5 = 0$, $-9x^2 + 7x + 5 = 0$,

$x^2 + 2x = 0$, $2x^2 = 0$

General form :

$$ax^2 + bx + c = 0$$

Where, $a, b, c \in C$ and $a \neq 0$

3.1 Roots of a Quadratic Equation

The values of variable x which satisfy the quadratic equation is called as Roots (also called solutions or zeros) of a Quadratic Equation.

4. Solution of Quadratic Equation

4.1 Factorization Method :

Let $ax^2 + bx + c = a(x-\alpha)(x-\beta) = 0$

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation.

Hence factorize the equation and equating each to zero gives roots of equation.

e.g. $3x^2 - 2x - 1 = 0 \equiv (x-1)(3x+1) = 0$

$$x = 1, \frac{1}{3}$$

4.2 Hindu Method (Sri Dharacharya Method) :

By completing the perfect square as

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Adding and substracting $\left(\frac{b}{2a}\right)^2$

$$\Rightarrow \left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

Which gives, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hence the Quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note : Every quadratic equation has two and only two roots.

5. Nature of Roots

In Quadratic equation $ax^2 + bx + c = 0$, the term $b^2 - 4ac$ is called discriminant of the equation, which plays an important role in finding the nature of the roots. It is denoted by Δ or D.

(A) Suppose a, b, c $\in$ R and a $\neq$ 0 then

(i) If D > 0 $\Rightarrow$ Roots are Real and unequal

(ii) If D = 0 $\Rightarrow$ Roots are Real and equal and each equal to $-b/2a$

(iii) If D < 0 $\Rightarrow$ Roots are imaginary and unequal or complex conjugate.

(B) Suppose a, b, c $\in$ Q, a $\neq$ 0 then

(i) If D > 0 and D is perfect square

$\Rightarrow$ Roots are unequal and Rational

(ii) If D > 0 and D is not perfect square

$\Rightarrow$ Roots are irrational and unequal

5.1 Conjugate Roots :

The Irrational and complex roots of a quadratic equation are always occurs in pairs. Therefore (a, b, c,$\in$Q)

If One Root	then	Other Root
$\alpha + i\beta$		$\alpha - i\beta$
$\alpha + \sqrt{\beta}$		$\alpha - \sqrt{\beta}$

6. Sum and Product of Roots

If α and β are the roots of quadratic equation $ax^2 + bx + c = 0$, then,

(i) Sum of Roots

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

(ii) Product of Roots

$$P = \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

e.g. In equation

$$3x^2 + 4x - 5 = 0$$

Sum of roots $S = -\frac{4}{3}$,

Product of roots $P = -\frac{5}{3}$

6.1 Relation between Roots and Coefficients

If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are and then :

(i) $(\alpha - \beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \pm\frac{\sqrt{b^2 - 4ac}}{a} = \frac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2 - 2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$$= -\frac{b\sqrt{b^2 - 4ac}}{a^2} = \frac{\pm\sqrt{D}}{a}$$

(iv) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = -\frac{b(b^2 - 3ac)}{a^3}$

(v) $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$$= \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\left\{(\alpha+\beta)^2 - \alpha\beta\right\}$$

$$= \frac{(b^2 - ac)\sqrt{b^2 - 4ac}}{a^3}$$

(vi) $\alpha^4 + \beta^4 = \left\{(\alpha+\beta)^2 - 2\alpha\beta\right\}^2 - 2\alpha^2\beta^2$

$$= \left(\frac{b^2 - 2ac}{a^2}\right) - 2\frac{c^2}{a^2}$$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2)$

$$= \frac{-b(b^2 - 2ac)\sqrt{b^2 - 4ac}}{a^4}$$

(viii) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta$

(ix) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

(x) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha + \beta)$

(xi) $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2 = \frac{\alpha^4 + \beta^4}{\alpha^4\beta^4} = \frac{(\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2}$

7. Formation of an Equation with given Roots

A quadratic equation whose roots are α and β is given by

$$(x - \alpha)(x - \beta) = 0$$

$\therefore$ $x^2 - \alpha x - \beta x + \alpha\beta = 0$

$\therefore$ $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

i.e x^2 – (sum of Roots)x + Product of Roots = 0

$\therefore$ $x^2 - Sx + P = 0$

7.1 Equation in terms of the Roots of another Equation

If are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (Replace x by – x)

(ii) $1/\alpha, 1/\beta \Rightarrow cx^2 + bx + a = 0$ (Replace x by 1/x)

(iii) α^n, β^n ; $n \in N$ $a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ (Replace x by $x^{1/n}$)

(iv) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0$ (Replace x by x/k)

(v) $k + \alpha, k + \beta \Rightarrow a(x-k)^2 + b(x-k) + c = 0$
(Replace x by (x–k))

(vi) $\frac{\alpha}{k}, \frac{\beta}{k}$ $k^2ax^2 + kbx + c = 0$
(Replace x by kx)

(vii) $\alpha^{1/n}, \beta^{1/n}; n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$
(Replace x by x^n)

7.2 Symmetric Expressions

The symmetric expressions of the roots , of an equation are those expressions in and , which do not change by interchanging and . To find the value of such an expression, we generally express that in terms of and .

Some examples of symmetric expressions are–

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^2 + \alpha\beta + \beta^2$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta}$ (iv) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(v) $\alpha^2\beta + \beta^2\alpha$ (vi) $\left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\beta}{\alpha}\right)^2$

(vii) $\alpha^3 + \beta^3$ (viii) $\alpha^4 + \beta^4$

8. Roots Under Particular Cases

For the quadratic equation $ax^2 + bx + c = 0$

(i) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign

(ii) If $c = 0$ $\Rightarrow$ one root is zero other is $-b/a$

(iii) If $b = c = 0$ $\Rightarrow$ both root are zero

(iv) If $a = c$ $\Rightarrow$ roots are reciprocal to each other

(v) If $\left.\begin{matrix} a > 0 \ \ c < 0 \\ a < 0 \ c > 0 \end{matrix}\right\} \Rightarrow$ Roots are of opposite signs

(vi) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ Both roots are negative.

(vii) $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ Both roots are positive.

(viii) If sign of a = sign of b $\neq$ sign of c
$\Rightarrow$ Greater root in magnitude is negative.

(ix) If sign of b = sign of c $\neq$ sign of a
$\Rightarrow$ Greater root in magnitude is positive.

(x) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a.

(xi) If $a = b = c = 0$ then equation will become an identity and will be satisfy by every value of x.

9. Condition for Common Roots

9.1 Only One Root is Common : Let α be the common root of quadratic equations

$a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ then

$\therefore$ $a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

By Cramer's rule :

$$\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha^2}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}, \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \alpha \neq 0.$$

$\therefore$ The condition for only one Root common is $(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

9.2 Both roots are common : Then required conditions is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Note : Two different quadratic equation with rational coefficient cannot have single common root which is complex or irrational, as imaginary and surd roots always occur in pair.

10. Nature of the Factors of the Quadratic Expression

The nature of factors of the quadratic expression $ax^2 + bx + c$ is the same as the nature of roots of the corresponding quadratic equation $ax^2 + bx + c = 0$ (a, b, c, $\in$R). Thus the factors of the expression are:

(i) Real and different, if $b^2 - 4ac > 0$.

(ii) Rational and different, if $b^2 - 4ac$ is a perfect square where (a, b, c, $\in$Q).

(iii) Real and equal , if $b^2 - 4ac = 0$.

(iv) Imaginary, if $b^2 - 4ac < 0$.

eg. The factors of $x^2 - x + 1$ are -

Sol. The factors of $x^2 - x + 1$ are imaginary because

$b^2 - 4ac = (-1)^2 - 4(1)(1)$

$= 1 - 4 = -3 < 0$

11. Maximum & Minimum Value of Quadratic Expression

In a Quadratic Expression $ax^2 + bx + c$

(i) If a > 0 Quadratic expression has least value at $x = -\frac{b}{2a}$. This least value is given by $\frac{4ac - b^2}{4a} = -\frac{D}{4a}$

(ii) If a < 0, Quadratic expression has greatest value s at $x = -\frac{b}{2a}$. This greatest value is given by $\frac{4ac - b^2}{4a} = -\frac{D}{4a}$

12. Sign of the Quadratic Expression

Let $y = ax^2 + bx + c \quad (a \neq 0)$

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right] \qquad ...(1)$$

Where $D = b^2 - 4ac$ is the Discriminant of the quadratic equation $ax^2 + bx + c = 0$

Case 1.

D > 0 : Suppose the roots of $ax^2 + bx + c = 0$ are α and β and $\alpha > \beta$ (say).

α, β are real and distinct.

Then $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

Clearly $(x - \alpha)(x - \beta) > 0$ for $x < \beta$ and $x < \alpha$ since both factors are of the same sign and

$(x - \alpha)(x - \beta) < 0$ for $\alpha > x > \beta$

For $x = \beta$ or $x = \alpha$, $(x - \alpha)(x - \beta) = 0$

$\therefore$ If a > 0, then $ax^2 + bx + c > 0$ for all x outside the interval $[\beta, \alpha]$ and is negative for all x in (β, α).

If a < 0, then its viceversa.

Case 2.

D = 0 then from (1)

$$ax^2 + bx + c = a\left(x + \frac{b^2}{2a}\right)^2$$

$\therefore$ $\forall\, x \neq -\frac{b}{2a}$, the quadratic expression takes on values of the same sign as a;

If $x = -b/2a$ then $ax^2 + bx + c = 0$.

$\therefore$ If D = 0, then

(i) $ax^2 + bx + c > 0$ has a solution any $x \neq -\frac{b}{2a}$ if a > 0 and has no solution if a < 0;

(ii) $ax^2 + bx + c < 0$ has a solution any $x \neq -\frac{b}{2a}$ if a < 0 and has no solution if a > 0;

(iii) $ax^2 + bx + c \geq 0$ has any x as a solution if a > 0 and the unique solution $x = -\frac{b}{2a}$, if a < 0;

(iv) $ax^2 + bx + c \leq 0$ has any x as a solution if a < 0 and $x = -\frac{b}{2a}$, if a > 0;

Case 3.

D < 0 then from (1)

(i) if a > 0, then $ax^2 + bx + c > 0$ for all x;

(ii) if a < 0, then $ax^2 + bx + c < 0$ for all x.

eg. The sign of $x^2 + 2x + 3$ is positive for all $x \in R$, because here $b^2 - 4ac = 4 - 12 = -8 < 0$ and $a = 1 > 0$.

eg. The sign of $3x^2 + 5x - 8$ is negative for all $x \in R$ because here $b^2 - 4ac = 25 - 96 = -71 < 0$ and $a = -3 < 0$

12.1 Graph of Quadratic Expression :

Consider the expression $y = ax^2 + bx + c$, $a \neq 0$ and $a, b, c \in R$ then the graph between x, y is always a parabola. If a > 0 then the shape of the parabola in concave upward and if a < 0 then the shape of the parabola is concave downwards.

There is only 6 possible graph of a Quadratic expression as given below :

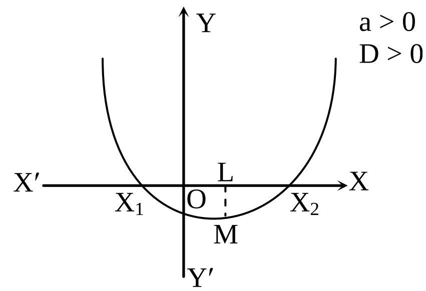
Case - I When a > 0

(i) If D > 0

Roots are real and different (X_1 and X_2)

Minimum value $LM = \frac{4ac - b^2}{4a}$ at $x = OL = -b/2a$

y is positive for all x out side interval $[x_1, x_2]$ and is negative for all x inside (x_1, x_2)

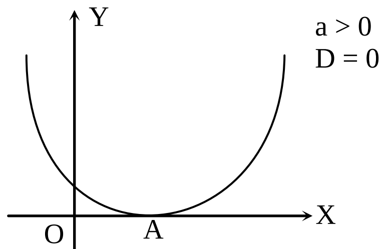


(ii) If D = 0

Roots are equal (OA)

Min. value = 0 at x = OA = –b/2a

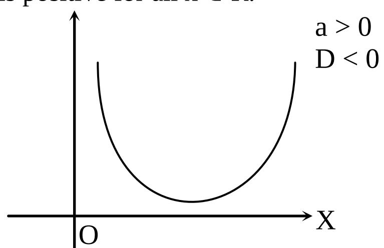
y > 0 for all $x \in \left\{R - \frac{-b}{2a}\right\}$



(iii) If D < 0

Roots are complex conjugate

y is positive for all $x \in R$.



Case- II When a < 0

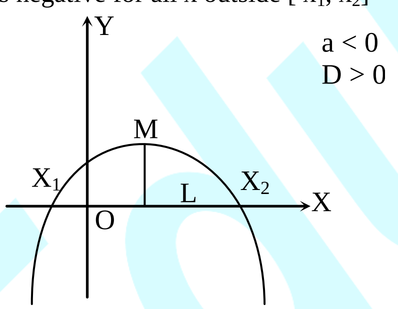
(i) If D > 0

Roots are real and different (x_1 and x_2)

Max. value = LM = $\frac{4ac - b^2}{4a}$ at x = OL = –b/2a

y is positive for all x inside (x_1, x_2) and

y is negative for all x outside [x_1, x_2]

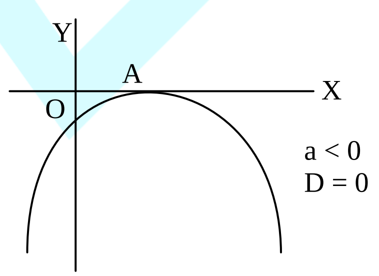


(ii) If D = 0

Roots are equal = OA

Max. value = 0 at x = OA = –b/2a

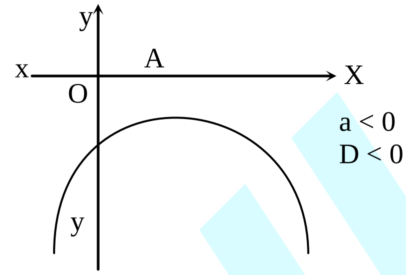
y is negative for all $x \in \left\{R - \frac{-b}{2a}\right\}$



(iii) If D < 0

Roots are complex conjugate

y is negative for all $x \in R$



13. Quadratic Expression in two Variables

The general form of a quadratic expression in two variable x & y is

$$ax^2 + 2hxy + by^2 + 2\,gx + 2\,fy + c$$

The condition that this expression may be resolved into two linear rational factors is

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0$ and $h^2 - ab > 0$

This expression is called discriminant of the above quadratic expression.

14. Some Important Points

(i) Every equation of n^{th} degree ($n \geq 1$) has exactly n roots and if the equation has more than n roots, it is an identity.

(ii) If α is a root of the equation f (x) = 0 then the polynomial f (x) is exactly divisible by

$(x–\alpha)$ or $(x – \alpha)$ is a factor of f (x)

(iii) If quadratic equations $a_1 x^2 + b_1 x + c_1 = 0$ and $a_2 x^2 + b_2 x + c_2 = 0$ are in the same ratio $\left(\text{i.e.} \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}\right)$ then

$$\frac{b_1^2}{b_2^2} = \frac{a_1c_1}{a_2c_2}$$

(iv) If one root is k times the other root of quadratic equation $a_1 x^2 + b_1 x + c_1 = 0$ then

$$\frac{(k+1)^2}{k} = \frac{b^2}{ac}$$

(v) Quadratic equations containing modulas sign are solved considering both positive and negative values of the quantity containing modulus sign. Finally the roots of the given equation will be those values among the values of the variable so obtained which satisfy the given equation.