Indefinite Integration

1. Integration of a Function

Integration is a reverse process of differentiation. The **integral or primitive** of a function f(x) with respect to x is that function (x) whose derivative with respect to x is the given function f(x). It is expressed symbolically as -

$$\int f(x)\,dx = \phi(x)$$

Thus

$$\int f(x)\,dx = \phi(x) \Leftrightarrow \frac{d}{dx}[\phi(x)] = f(x)$$

The process of finding the **integral** of a function is called Integration and the given function is called **Integrand**. Now, it is obvious that the operation of integration is inverse operation of differentiation. Hence integral of a function is also named as **anti-derivative** of that function.

Further we observe that-

$$\left.\begin{aligned}\frac{d}{dx}(x^2) &= 2x\\ \frac{d}{dx}(x^2+2) &= 2x\\ \frac{d}{dx}(x^2+k) &= 2x\end{aligned}\right\} \Rightarrow \int 2x\,dx = x^2 + \text{constant}$$

So we always add a constant to the integral of function, which is called the **constant of Integration**. It is generally denoted by c. Due to presence of this constant such an integral is called an **Indefinite integral**.

2. Basic Theorems on Integration

If f(x), g(x) are two functions of a variable x and k is a constant, then-

(i) $\int k\ f(x)\,dx = k\int f(x)\,dx.$

(ii) $\int [f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(iii) $d/dx(\int f(x)\,dx) = f(x)$

(iv) $\int \left(\frac{d}{dx} f(x)\right) dx = f(x)$

3. Standard Integrals

The following integrals are directly obtained from the derivatives of standard functions.

i. $\int 0\,.\,dx = c$

ii. $\int 1.dx = x + c$

iii. $\int k\,.dx = kx + c\ (k \in R)$

iv. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ (n–1)$

v. $\int \frac{1}{x}\,dx = \log_e x + c$

vi. $\int e^x\,dx = e^x + c$

vii. $\int a^x\,dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$

viii. $\int \sin x\,dx = -\cos x + c$

ix. $\int \cos x\,dx = \sin x + c$

x. $\int \tan x\,dx = \log \sec x + c = -\log \cos x + c$

xi. $\int \cot x\,dx = \log \sin x + c$

xii. $\int \sec x\,dx = \log(\sec x + \tan x) + c$

$= -\log(\sec x - \tan x) + c$

$= \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

xiii. $\int \operatorname{cosec} x\,dx = -\log(\operatorname{cosec} x + \cot x) + c$

$= \log(\operatorname{cosec} x - \cot x) + c = \log \tan\left(\frac{x}{2}\right) + c$

xiv. $\int \sec x \tan x\,dx = \sec x + c$

xv. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

xvi. $\int \sec^2 x\,dx = \tan x + c$

xvii. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

xviii. $\int \sinh x\,dx = \cosh x + c$

xix. $\int \cosh x\,dx = \sinh x + c$

xx. $\int \operatorname{sech}^2 x\,dx = \tanh x + c$

xxi. $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$

xxii. $\int \operatorname{sech} x\ \tanh x\,dx = -\operatorname{sech} x + c$

xxiii. $\int \operatorname{cosech} x \coth x = -\operatorname{cosech} x + c$

xxiv. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

xxv. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$

xxvi. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left(\frac{x+a}{x-a}\right) + c$

xxvii. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$= -\cos^{-1}\left(\frac{x}{a}\right) + c$

xxviii. $\int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$

$= \log(x + \sqrt{x^2+a^2}) + c$

xxix. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$

$= \log(x + \sqrt{x^2-a^2}) + c$

xxx. $\int \sqrt{a^2-x^2}\,dx$

$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}.\sin^{-1}\frac{x}{a} + c$

xxxi. $\int \sqrt{x^2+a^2}\,dx$

$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}.\sinh^{-1}\frac{x}{a} + c$

xxxii. $\int \sqrt{x^2-a^2}\,dx$

$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}.\cosh^{-1}\frac{x}{a} + c$

xxxiii. $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

xxxiv. $\int e^{ax}\sin bx\,dx$

$= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx - \tan^{-1}\left(\frac{b}{a}\right)\right\} + c$

xxxv. $\int e^{ax}\cos bx\,dx$

$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx - \tan^{-1}\frac{b}{a}\right\} + c$

4. Methods of Integration

When integration can not be reduced into some standard form then integration is performed using following methods-

(i) Integration by substitutions

(ii) Integration by parts

(iii) Integration of rational functions

(iv) Integration of irrational functions

(v) Integration of trigonometric functions

4.1 INTEGRATION BY SUBSTITUTION:

Generally we apply this method in the following two cases.

(i) When Integrand is a function of function -

i.e. $\int f[\phi(x)]\,\phi'(x)\,dx$

Here we put $\phi(x) = t$ so that $\phi'(x)\,dx = dt$ and in that case the integrand is reduced to $\int f(t)\,dt$.

Note:

In this method the integrand is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

(ii) When integrand is the product of two factors such that one is the derivative of the other i.e. I = $\int f'(x)\,f(x)\,dx$.

In this case we put f(x) = t and convert it into a standard integral.

(iii) Integral of a function of the form f (ax + b).

Here we put ax + b = t and convert it into standard integral. Obviously if $\int f(x)\,dx = \phi(x)$, then

$$\int f(ax+b)\,dx = \frac{1}{a}\phi(ax+b)$$

(iv) Standard form of Integrals:

(a) $\int \frac{f'(x)}{f(x)}dx = \log[f(x)] + c$

(b) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$

(provided $n \neq -1$)

(c) $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

(v) Integral of the form

$$\int \frac{dx}{a\sin x + b\cos x}$$

putting a = r cos θ and b = r sin θ. we get

$$I = \int \frac{dx}{r\sin(x+\theta)} = \frac{1}{r}\int \text{cosec}(x+\theta)\,dx.$$

$$= \frac{1}{r}\log\tan(x/2 + \theta/2) + c$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\tan(x/2 + 1/2\tan^{-1} b/a) + c$$

(vi) Standard Substitutions : Following standard substitutions will be useful-

Integrand form	Substitution
(i) $\sqrt{a^2-x^2}$ or	$x = a\sin\theta$ or
$\frac{1}{\sqrt{a^2-x^2}}$	$x = a\cos\theta$
(ii) $\sqrt{x^2+a^2}$ or	$x = a\tan\theta$ or
$\frac{1}{\sqrt{x^2+a^2}}$	$x = a\cot\theta$ or $x = a\sinh\theta$
(iii) $\sqrt{x^2-a^2}$ or	$x = a\sec\theta$ or
$\frac{1}{\sqrt{x^2-a^2}}$	$x = a\,\text{cosec}\,\theta$ or $x = a\cosh\theta$
(iv) $\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$	$x = a\tan^2\theta$
or $\sqrt{x(a+x)}$ or $\frac{1}{\sqrt{x(a+x)}}$	
(v) $\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$	
or $\sqrt{x(a-x)}$	$x = a\sin^2\theta$
or $\frac{1}{\sqrt{x(a-x)}}$	
(vi) $\sqrt{\frac{x}{x-a}}$ or	$x = a\sec^2\theta$
$\sqrt{\frac{x-a}{x}}$ or	
$\sqrt{x(x-a)}$ or	
(vii) $\sqrt{\frac{a-x}{a+x}}$ or	$x = a\cos 2\theta$
$\sqrt{\frac{a+x}{a-x}}$	
(viii) $\sqrt{\frac{x-\alpha}{\beta-x}}$ or	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
$\sqrt{(x-\alpha)(\beta-x)}$	
$(\beta > \alpha)$	

4.2 INTEGRATION BY PARTS :

4.2.1 If u and v are two functions of x, then

$$\int (u.v)\,dx = u\left(\int v\,dx\right) - \int\left(\frac{du}{dx}\right).\left(\int v\,dx\right)dx.$$

i.e. Integral of the product of two functions = first function × integral of second function – ∫ [(derivative of first) × (Integral of second)]

Note :

(i) From the first letter of the words inverse circular, logarithmic, Algebraic, Trigonometric, Exponential functions, we get a word **ILATE**. Therefore first arrange the functions in the order according to letters of this word and then integrate by parts.

(ii) For the integration of Logarithmic or Inverse trigonometric functions alone, take unity (1) as the second function.

4.2.2 If the integral is of the form $\int e^x [f(x) + f'(x)]dx$, then by breaking this integral into two integrals, integrate one integral by parts and keep other integral as it is, By doing so, we get-

$$\int e^x[f(x)+f'(x)]dx = e^x f(x) + c$$

4.2.3 If the integral is of the form $\int [x f'(x) + f(x)]dx$ then by breaking this integral into two integrals integrate one integral by parts and keep other integral as it is, by doing so, we get

$$\int [x f'(x) + f(x)]dx = x f(x) + c$$

4.3 Integration of Rational functions:

4.3.1 When denominator can be factorized (using partial fraction) :

Let the integrand is of the form $\frac{f(x)}{g(x)}$, where both f(x) and g(x) are polynomials. If degree of f(x) is greater than degree of g(x) then first divide f(x) by g(x) till the degree of the remainder becomes less than the degree of g(x). Let Q(x) is the quotient and R(x), the remainder then

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$$

Now in R(x)/g(x), factorize g(x) and then write partial fractions in the following manner-

(i) For every non repeated linear factor in the denominator, write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

(ii) For repeated linear factors in the denominator, write-

$$\frac{1}{(x-a)^3(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$$

(iii) For every non repeated quadratic factor in the denominator, write

$$\frac{1}{(ax^2+bx+c)(x-d)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x-d}$$

Note :

(i) If integrand is of the form $\frac{1}{(x+a)(x+b)}$ then use the following method for obtaining partial fractions-

Here $\frac{1}{(x+a)(x+b)} = \frac{1}{(a-b)}\left[\frac{a-b}{(x+a)(x+b)}\right]$

$$= \frac{1}{(a-b)}\left[\frac{(x+a)-(x+b)}{(x+a)(x+b)}\right]$$

$$= \frac{1}{(a-b)}\left[\frac{1}{x+b} - \frac{1}{x+a}\right]$$

(ii) If integrand is of the form $\frac{x}{(x+a)(x+b)}$ then

$$\frac{x}{(x+a)(x+b)} = \frac{1}{b-a}\left[\frac{(b-a)x}{(x+a)(x+b)}\right]$$

$$= \frac{1}{b-a}\left[\frac{b(x+a)-a(x+b)}{(x+a)(x+b)}\right]$$

$$= \frac{1}{b-a}\left[\frac{b}{x+b} - \frac{a}{x+a}\right]$$

4.3.2 When denominator can not be factorised:

In this case integral may be in the form

(i) $\int \frac{dx}{ax^2+bx+c}$, **(ii)** $\int \frac{(px+q)}{ax^2+bx+c}dx$

Method:

(i) Here taking coefficient of x^2 common from denominator, write -

$$x^2 + (b/a)x + c/a = (x + b/2a)^2 - \frac{b^2-4ac}{4a^2}$$

Now the integrand so obtained can be evaluated easily by using standard formulas.

(ii) Here suppose that px + q = A [diff. coefficient of $(ax^2 + bx + c)$] + B

= A (2ax + b) + B ...(1)

Now comparing coefficient of x and constant terms.

we get A = p/2a, B = q – (pb/2a)

$$\therefore I = P/2a \int \frac{2ax+b}{ax^2+bx+c}dx + \left(q - \frac{pb}{2a}\right)\int \frac{dx}{ax^2+bx+c}$$

Now we can integrate it easily.

4.3.3 Integration of rational functions containing only even powers of x.

To find integral of such functions, first we divide numerator and denominator by x^2, then express numerator as d(x ± 1/x) and denominator as a function of (x ± 1/x).

4.4 Integration of irrational functions :

If anyone term in Nr or Dr is irrational then it is made rational by suitable substitution. Also if integral is of the form-

$$\int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\, dx$$

then we integrate it by expressing

$ax^2 + bx + c = (x + \alpha)^2 + \beta$

Also for integrals of the form

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int (px+q)\sqrt{ax^2+bx+c}\, dx.$$

First we express px + q in the form

$px + q = A\left\{\frac{d}{dx}(ax^2+bx+c)\right\} + B$ and then proceed as usual with standard form.

4.5 Integration of Trigonometric functions :

Here we shall study the methods for evaluation of following types of integrals.

I. **(i)** $\int \frac{dx}{a+b\sin^2 x}$

(ii) $\int \frac{dx}{a+b\cos^2 x}$

(iii) $\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}$

(iv) $\int \frac{dx}{(a\sin x + b\cos x)^2}$

Method :

Divide numerator and Denominator by $\cos^2 x$ in all such type of integrals and then put tan x = t.

II. **(i)** $\int \frac{dx}{a + b\cos x}$

(ii) $\int \frac{dx}{a + b\sin x}$

(iii) $\int \frac{dx}{a\cos x + b\sin x}$

(iv) $\int \frac{dx}{a\sin x + b\cos x + c}$

Method :

In such types of integrals we use following formulae for sin x and cos x in terms of tan (x/2).

$$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}, \cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

and then take tan(x/2) = t and integrate another method for evaluation of integral

(iii) put a = r cos α, b = r sin α, then

$$I = \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)}$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\alpha)\,dx .$$

$$= \frac{1}{r}\log\tan(x/2 + \alpha/2) + c$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{b}{a}\right) + c$$

III. $\int \frac{p\sin x + q\cos x}{a\sin x + b\cos x}dx$

$\int \frac{p\sin x}{a\sin x + b\cos x}dx$

$\int \frac{q\cos x}{a\sin x + b\cos x}dx$

For their integration, we first express Nr. as follows-

Nr = A (Dr) + B (derivative of Dr.)

Then integral = Ax + B log (Dr) + C

5. Some Integrates of Different Expression of e^x

(i) $\int \frac{ae^x}{b+ce^x}dx$ [put $e^x = t$]

(ii) $\int \frac{1}{1+e^x}dx$ [Multiply and divide by e^{-x} and put e^{-x}=t]

(iii) $\int \frac{1}{1-e^x}dx$ [Multiply and divide by e^{-x} and put e^{-x}=t]

(iv) $\int \frac{1}{e^x - e^{-x}}dx$ [Multiply and divide by e^x]

(v) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ $\left[\frac{f'(x)}{f(x)}\text{form}\right]$

(vi) $\int \frac{e^x+1}{e^x-1}dx$ [Multiply and divide by$e^{-x/2}$]

(vii) $\int \left(\frac{e^x - e^{-x}}{e^x - e^{-x}}\right)^2 dx$ [Integrand = $\tanh^2 x$]

(viii) $\int \left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2 dx$ [Integrand = $\coth^2 x$]

(ix) $\int \frac{1}{(e^x + e^{-x})^2}dx$ [Integrand = 1/4 sech^2 x]

(x) $\int \frac{1}{(e^x - e^{-x})^2}dx$ [Integrand=1/4cosech^2 x]

(xi) $\int \frac{1}{(1+e^x)(1-e^{-x})}dx$ [Multiply and divide by e^x and put e^x = t]

(xii) $\int \frac{1}{\sqrt{1-e^x}}dx$ [Multiply and divide by$e^{-x/2}$]

(xiii) $\int \frac{1}{\sqrt{1+e^x}}dx$ [Multiply and divide by $e^{-x/2}$]

(xiv) $\int \frac{1}{\sqrt{e^x-1}}dx$ [Multiply & divide by $e^{-x/2}$]

(xv) $\int \frac{1}{\sqrt{2e^x-1}}dx$ [Multiply and divide by $\sqrt{2}\ e^{-x/2}$]

(xvi) $\int \sqrt{1-e^x}\,dx$ [Integrand = $(1-e^x)/\sqrt{1-e^x}$]

(xvii) $\int \sqrt{1+e^x}\,dx$ [Integrand =$(1+e^x)/\sqrt{1+e^x}$]

(xviii) $\int \sqrt{e^x-1}\,dx$ [Integrand = $(e^x-1)/\sqrt{e^x-1}$]

(xix) $\int \sqrt{\frac{e^x+a}{e^x-a}}dx$ [Integrand = $(e^x+a)/\sqrt{e^{2x}-a^2}$