Hyperbolic Function

1. Definition

If x is a variable number (real or complex), then expression $\frac{e^x - e^{-x}}{2}$ is denoted **sinh x**, and it is read as **'hyperbolic sine x'**. Similarly expression $\frac{e^x + e^{-x}}{2}$ is denoted by **cosh x** and it is read as **'hyperbolic cosine x'.** Similar to trigonometrical functions the remaining four hyperbolic functions can be defined in terms of sinh x and cosh x. Thus hyperbolic functions are defined as follows-

(i) $\sinh x = \frac{e^x - e^{-x}}{2}$

(ii) $\cosh x = \frac{e^x + e^{-x}}{2}$

(iii) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(iv) $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(v) $\text{cosech } x = \frac{2}{e^x - e^{-x}}$

(vi) $\text{sech } x = \frac{2}{e^x + e^{-x}}$

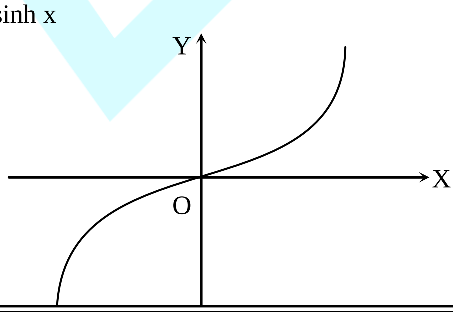
Note : $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$

2. Significance of the Name

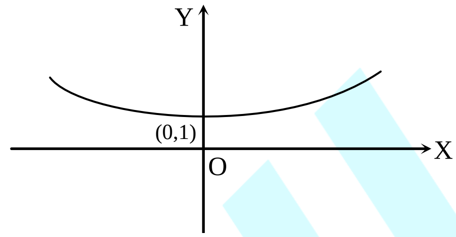
We know that parametric coordinates of any point on the unit circle $x^2 + y^2 = 1$ is $(\cos\theta, \sin\theta)$, so that these functions are called circular functions and coordinates of any point on unit Hyperbola $x^2 - y^2 = 1$ is $\left(\frac{e^\theta + e^{-\theta}}{2}, \frac{e^\theta - e^{-\theta}}{2}\right)$ i.e., (cosh , sinh). It means that the relation which exists amongst cos θ, sin θ and unit circle, that relation also exist amongst cosh θ, sinh θ and unit Hyperbola. Because of this reason these functions are called as **Hyperbolic Functions**.

3. Graph of Real Hyperbolic Functions

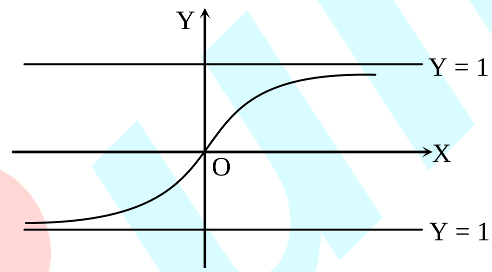
(i) sinh x



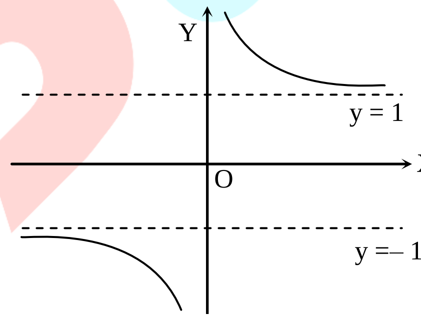
(ii) cosh x



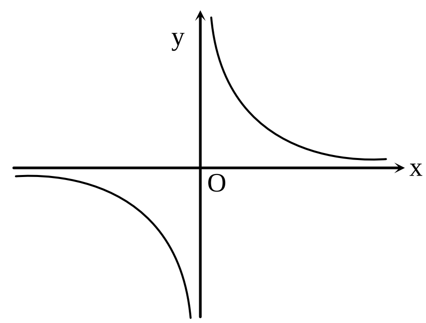
(iii) tanh x



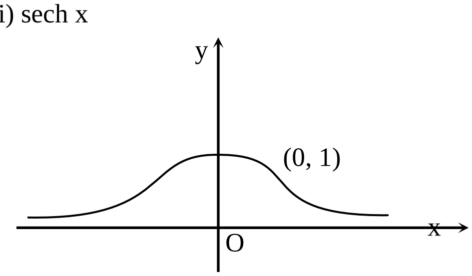
(iv) coth x



(v) cosech x



(vi) sech x



4. Domain & Range of Hyperbolic Functions

Let x is any real number.

Function	Domain	Range
sinh x	R	R
cosh x	R	[1, ∞)
tanh x	R	(– 1,1)
coth x	R_0	R– [–1,1]
cosech x	R_0	R_0
sech x	R	(0,1]

5. Formulae for Hyperbolic Functions

The following formulae can easily be established directly from above definitions-

A. Reciprocal Formulae :

(i) $\text{cosech } x = \dfrac{1}{\sinh x}$

(ii) $\text{sech } x = \dfrac{1}{\cosh x}$

(iii) $\coth x = \dfrac{1}{\tanh x}$

(iv) $\tanh x = \dfrac{\sinh x}{\cosh x}$

(v) $\coth x = \dfrac{\cosh x}{\sinh x}$

B. Square Formulae :

(i) $\cosh^2 x - \sinh^2 x = 1$

(ii) $\text{sech}^2 x + \tanh^2 x = 1$

(iii) $\coth^2 x - \text{cosech}^2 x = 1$

(iv) $\cosh^2 x + \sinh^2 x = \cosh 2x$

C. Expansion Formulae:

(i) sinh (x ± y) = sinh x cosh y cosh x sinhy

(ii) cosh (x ± y) = cosh x cosh y sinh x sinhy

(iii) $\tanh (x \pm y) = \dfrac{\tan x \pm \tanh y}{1 \pm \tanh + \tanh y}$

D. (i) $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

(ii) $\sinh x - \sinh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

(iii) $\cosh x + \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$

(iv) $\cosh x - \cosh y = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

E. (i) 2 sinh x cosh y = sinh (x+ y) + sinh (x– y)

(ii) 2 cosh x sinh y = sinh (x + y) – sinh (x– y)

(iii) 2 cosh x cosh y = cosh (x+ y) + cosh (x– y)

(iv) 2 sinh x sinh y = cosh (x+ y) – cosh (x– y)

F. (i) $\sinh 2x = 2 \sinh x \cosh x = \dfrac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

(iii) $2\cosh^2 x = \cosh 2x + 1$

(iv) $2\sinh^2 x = \cosh 2x - 1$

(v) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

G. (i) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(ii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(iii) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

H. (i) $\cosh x + \sinh x = e^x$

(ii) $\cosh x - \sinh x = e^{-x}$

(iii) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

6. To Express any one Hyperbolic Function in Terms of Others

Since $\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \sinh x = \sqrt{\cosh^2 x - 1}$ ...(1)

$\Rightarrow \sinh x = \sqrt{\dfrac{1-\text{sech}^2 x}{\text{sech} x}}$ ...(2)

$\Rightarrow \sinh x = \dfrac{\tanh x}{\sqrt{1-\tanh^2 x}}$ ...(3)

$\Rightarrow \sinh x = \dfrac{1}{\sqrt{\coth^2 x - 1}}$ ...(4)

7. Expansions of Hyperbolic Functions

(i) $\sinh x = \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + ...$

(ii) $\cosh x = \dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^5}{6!} + ...$

(iii) $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = x - \dfrac{x^3}{3} + 2x^5 - \dfrac{17}{315}x^7 + ...$

The expansion of coth x, cosech x does not exist because coth (0) = ∞, cosech (o) = ∞

8. Relation between Hyperbolic and Circular Functions

We have from Euler formulae

$e^{ix} = \cos x + i \sin x$ ...(1)

and $e^{-ix} = \cos x - i \sin x$ ...(2)

adding (1) & (2)

$\Rightarrow \cos x = \dfrac{e^{ix} + e^{-x}}{2}$

Subtracting (2) from (1)

$\Rightarrow \sin x = \dfrac{e^{-x} - e^{-x}}{2i}$

Replacing x by ix in these values, we get

$$\cos(ix) = \frac{e^{-x}+e^{x}}{2} = \cosh x$$

$\therefore$ cos (ix) = cosh x

$$\sin(ix) = \frac{e^{-x}-e^{x}}{2i} = i\left(\frac{e^{x}-e^{-x}}{2}\right)$$

sin (ix) = i sinh x

Also $\tan(ix) = \dfrac{\sinh(ix)}{\cosh(ix)} = \dfrac{i\sinh x}{\cosh x}$

tan (ix) = i tanh x

Similarly replacing x by ix in the definitions of sinh x and cosh x, we get

$$\sinh(ix) = \frac{e^{ix}-e^{-ix}}{2} = i.\frac{e^{ix}-e^{-ix}}{2i} = i\sin x$$

$$\cosh(ix) = \frac{e^{ix}+e^{-ix}}{2i} = \cos x$$

Also $\tanh(ix) = \dfrac{\sin h(ix)}{\cosh(ix)} = \dfrac{i\sin x}{\cos x} = i\tan x$

Thus, we obtain the following relations between hyperbolic and trigonometrical functions.

(i) sin (ix) = i sinh x

(ii) cos (ix) = cosh x
sinh (ix) = i sin x
cosh (ix) = cos x
sinhx = – i sin (ix)
cosh x = cos (ix)
sin x = – i sinh (ix)
cos x = cosh (ix)

(iii) tan (ix) = i tanh x

(iv) cot (ix) = – i coth x
tanh (ix) = i tan x
coth (ix) = – i cot x
tanh x = – i tan (ix)
coth x = i cot (ix)
tan x = – i tanh (ix)
cot x = i coth (ix)

(v) sec (ix) = sech x

(vi) cosec (ix) = – i cosech x
sech (ix) = sec x
cosech (ix) = – i cosec x
sech x = sec (ix)
cosech x = i cosec (ix)
sec x = sech (ix)
cosec x = i cosech (ix)

Note :

For obtaining any formula given in $(5)^{th}$ article, use the following substitutions in the corresponding formula for trigonometric functions.

$\sin x \longrightarrow i\sinh x$, $\cos x \longrightarrow \cosh x$,
$\tan x \longrightarrow i\tanh x$
$\sin^2 x \longrightarrow -\sinh^2 x$, $\cos^2 x \longrightarrow \cosh^2 x$,
$\tan^2 x \longrightarrow -\tanh^2 x$

For example :

(i) For finding out the formula for cosh 2x in terms of tanh x, replace tan x by i tanh x and $\tan^2 x$ by $-\tanh^2 x$ in the following formula of trigonometric function of cos 2x :

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

We get

$$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$$

9. Period of Hyperbolic Functions

If for any function f(x) , f(x + T) = f(x) , then f(x) is called the **periodic function** and least positive value of T is called the **period of the function.**

sinh x = sinh (2 π i + x)

cosh x = cosh (2πi + x)

and tanh x = tanh (πi + x)

Therefore, the period of these functions are 2πi, 2i and i respectively. Also periods of cosech x, sech x and coth x are 2 π i, 2 π i and π i respectively.

Note :

Hyperbolic functions are neither periodic functions nor their curves are periodic, but they show the algebraic properties of periodic functions and having imaginary period.

10. Inverse Hyperbolic Functions

Definition : If sinh y = x, then y is called the **Inverse Hyperbolic sine of x** and it is written as $y = \sinh^{-1} x$. Similarly $\mathrm{cosech}^{-1} x$, $\cosh^{-1}x$, $\tanh^{-1} x$ etc. can be defined.

10.1 Domain & range of Inverse Hyperbolic Function :

Function	Domain	Range
$\sinh^{-1} x$	R	R
$\cosh^{-1} x$	[1,)	R
$\tanh^{-1} x$	(–1,1)	R
$\coth^{-1} x$	R– [–1,1]	R_0
$\mathrm{sech}^{-1} x$	(0,1]	R
$\mathrm{cosech}^{-1} x$	R_0	R_0

10.2 Relation between inverse Hyperbolic function and inverse circular function

Method : Let $\sinh^{-1} x = y$

x = sinh y
= – i sin (iy)
ix = sin (iy)
iy = $\sin^{-1}$ (ix)
y = – i $\sin^{-1}$(ix)

$\sinh^{-1} x = -i \sin^{-1} (ix)$

Therefore we get the following relations

(i) $\sinh^{-1} x = -i \sin^{-1} (ix)$

(ii) $\cosh^{-1} x = -i \cos^{-1} x$

(iii) $\tanh^{-1} x = -i \tan^{-1} (ix)$

(iv) $\coth^{-1} x = i \cot^{-1} (ix)$

(v) $\text{sech}^{-1} x = -i \sec^{-1} x$

(vi) $\text{cosech}^{-1} x = i\, \text{cosec}^{-1} (ix)$

10.3 To express any one inverse hyperbolic function in terms of the other inverse hyperbolic functions :

To express $\sinh^{-1} x$ in terms of the others

(i) Let $\sinh^{-1} x = y$

$\Rightarrow x = \sinh y$

$\Rightarrow \text{cosech } y = \dfrac{1}{x}$

$\Rightarrow y = \text{cosech}^{-1}\left(\dfrac{1}{x}\right)$

$\Rightarrow \sinh^{-1} x = \text{cosech}^{-1}\left(\dfrac{1}{x}\right)$

(ii) $\because \cosh y = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$

$\therefore y = \cosh^{-1} \sqrt{1+x^2}$

$\Rightarrow \sinh^{-1} x = \cosh^{-1} \sqrt{1+x^2}$

(iii) $\because \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{\sinh y}{\sqrt{1+\sinh^2 y}}$

$= \dfrac{x}{\sqrt{1+x^2}}$ $\quad \therefore y = \tanh^{-1} \dfrac{x}{\sqrt{1+x^2}}$

$\sinh^{-1} x = \tanh^{-1} x \dfrac{x}{\sqrt{1+x^2}}$

(iv) $\because \coth y = \dfrac{\sqrt{1+\sinh^2 y}}{\sinh y} = \dfrac{\sqrt{1+x^2}}{x}$

$\therefore y = \coth^{-1} \dfrac{\sqrt{1+x^2}}{x}$

$\Rightarrow \sinh^{-1} x = \coth^{-1} \dfrac{\sqrt{1+x^2}}{x}$

(v) $\because \text{sech } y = \dfrac{1}{\cosh y}$

$= \dfrac{1}{\sqrt{1+\sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$

$y = \text{sech}^{-1} \dfrac{1}{\sqrt{1+x^2}}$ ________

$\sinh^{-1} x = \text{sech}^{-1} \dfrac{1}{\sqrt{1+x^2}}$

(vi) Also $\sinh^{-1} x = \text{cosech}^{-1} (1/x)$

From the above it is clear that

$\coth^{-1} x = \tanh^{-1} (1/x)$

$\text{sech}^{-1} x = \cosh^{-1} (1/x)$

$\text{cosech}^{-1} = \sinh^{-1} (1/x)$

Note :

If x is real then all the above six inverse functions are single valued.

10.4 Relation between inverse hyperbolic functions and logarithmic functions

Method : Let $\sinh^{-1} x = y$

$\Rightarrow x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

$\Rightarrow e^{2y} - 2x\, e^y - 1 = 0$

$\Rightarrow e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$

But $e^y > 0$, $\forall$ y and $x < \sqrt{x^2+1}$

$\therefore e^y = x + \sqrt{x^2+1}$

$y = \log (x + \sqrt{x^2+1})$

$\sinh^{-1} x = \log (x + \sqrt{x^2+1})$

By the above method we can obtain the following relations between inverse hyperbolic functions and principal values of logarithmic functions :

(i) $\sinh^{-1} x = \log (x + \sqrt{x^2+1})$
$(-\infty < x < \infty)$

(ii) $\cosh^{-1} x = \log (x + \sqrt{x^2+1})$
$(x \geq 1)$

(iii) $\tanh^{-1} x = 1/2 \log \left(\dfrac{1+x}{1-x}\right)$ $\qquad |x| < 1$

(iv) $\coth^{-1} x = 1/2 \log \left(\dfrac{x+1}{x-1}\right)$ $\qquad |x| > 1$

(v) $\text{sech}^{-1} x = \log \left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$ $\qquad 0 < x\ 1$

(vi) $\text{cosech}^{-1} x = \log \left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$ $\qquad (x \neq 0)$

Note :

Formulae for values of $\text{cosech}^{-1} x$, $\text{sech}^{-1} x$ and $\coth^{-1} x$ may be obtained by replacing x by 1/x in the values of $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$ respectively.