Limit, Continuity & Differentiability

INDETERMINATE FORM

Some times we come across with some functions which donot have definite value corresponding to some particular value of the variable.

For example for the function

$f(x) = \frac{x^2-4}{x-2}$, $f(2) = \frac{4-4}{2-2} = \frac{0}{0}$

which cannot be determined. Such a form is called an Indeterminate form. Some other indeterminate forms are $0 \times \infty$, 0^0, 1^∞, $\infty - \infty$, ∞/∞, ∞°, 0/0.

LIMITS OF A FUNCTION

Let y = f (x) be a function of x and for some particular value of x say x = a, the value of y is indeterminate, then we consider the values of the function at the points which are very near to 'a'. If these values tend to a definite unique number ℓ as x tends to 'a' (either from left or from right) then this unique number ℓ is called the limits of f(x) at x = a and we write it as $\lim_{x \to a} f(x) = \ell$

Meaning of '$x \to a$' : Let x be a variable and a be a constant. If x assumes values nearer and nearer to 'a' then we can say ' x tends to a' and we write '$x \to a$'.

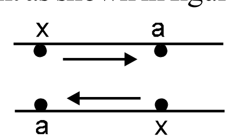
It should be noted that as $x \to a$ we have $x \neq a$.

By ' x tends to a' we mean that

(i) $x \neq a$

(ii) x assumes values nearer and nearer to 'a' and

(iii) we are not spacifying any manner in which x should approach to a.x may approach to a from left or right as shown in figure.



Ex.1 If $f(x) = \frac{x^2-9}{x-3}$ then $f(3) = \frac{0}{0}$, now when x tends to 3 from left or from right, it can be easily observed that the value of f(x) tend to 6 hence

$$\lim_{x \to 3} \cdot \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)}$$

$$= \lim_{x \to 3} (x+3) = 3 + 3 = 6$$

Ex.2 If $f(x) = x^3 + 1$ then $\lim_{x \to 1} f(x) = f(1) = 2$

Note : It is not necessary that if the value of a function at some point exists then its limit at that point must exist.

Ex.3 If f(x) = [x], then observing to its graph we find that f(2) = 2 but $\lim_{x \to 2} f(x)$ does not exist.

LEFT AND RIGHT LIMITS

If value of a function f(x) tend to a definite unique number when x tends to 'a' from left, then this unique number is called left hand limit (LHL) of f(x) at x = a and we can write it as

$f(a-0)$ or $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a-0} f(x)$

For evaluation

$f(a-0) = \lim_{h\to 0} f(a-h)$

Similarly, we can define right hand limit (RHL) of f(x) at x= a. In this case x tends to 'a' from right. We can write it as

$f(a+0)$ or $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a+0} f(x)$

For evaluation

$f(a+0) = \lim_{h\to 0} f(a+h)$

TO FIND LEFT/ RIGHT LIMIT

(i) For finding right hand limit of the function we write (x+h) in place of x while for left hand limit we write (x–h) in place of x.

(ii) We replace then x by a in the function so obtained.

(iii) Conclusively we find limit $h \to 0$

EXISTENCE OF LIMIT

The limit of a function at some point exists only when its left- hand limit and right hand limit at that point exist and are equal.

Thus $\lim_{x\to a} f(x)$ exists

$\Rightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \ell$

where ℓ is called the limit of the function.

Solved Examples

Ex.1 If $f(x) = \begin{cases} x^2+1 & x\ge 1 \\ 3x-1 & x<1 \end{cases}$ then the value of $\lim_{x\to 1}$ f(x) is -

(A) 1 (B) 2

(C) 3 (D) Does not exist

Sol. Left hand limit = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1)$

$= 3.1 - 1 = 2$

and Right hand limit = $\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} (x^2+1) = 1^2 + 1 = 2$

$\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$

So $\lim_{x\to 1} f(x) = 2$

Ex.2 The value of $\lim_{x\to 1} [x]$ is-

(A) 1 (B) 2

(C) 4 (D) Does not exist

Sol. Left hand limit = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} [x] = 0$ and

Right hand limit = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [x] = 1$

$\because \lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x)$

$\therefore$ limit does not exist.

Ex.3 The value of $\lim_{x\to 0} \frac{|x|}{x}$ is -

(A) 1 (B) 2

(C) 3 (D) Does not exist

Sol. Left hand limit = $\lim_{x\to 0^-} \frac{|x|}{x} = -1$ and

Right hand limit = $\lim_{x\to 0^+} \frac{|x|}{x} = 1$

$\because$ LHL $\ne$ RHL $\therefore$ Limit does not exist.

THEOREMS ON LIMITS

The following theorems are very helpful for evaluation of limits–

(i) $\lim_{x\to a} [k\,f(x)] = k \lim_{x\to a} f(x)$, where k is a constant

(ii) $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(iv) $\lim_{x\to a} [f(x).g(x)] = \lim_{x\to a} f(x). \lim_{x\to a} g(x)$

(v) $\lim\limits_{x\to a}[f(x)/g(x)] = [\lim\limits_{x\to a} f(x)]/[\lim\limits_{x\to a} g(x)]$ provided $g(x) \neq 0$

(vii) $\lim\limits_{x\to a}[f(x)+k] = \lim\limits_{x\to a} f(x) + k$ where k is a constant

(ix) If $f(x) \leq g(x)$ for all x, then $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$

(x) $\lim\limits_{x\to a}[f(x)]^{g(x)} = \{\lim\limits_{x\to a} f(x)\}^{\lim\limits_{x\to a} g(x)}$

(xi) $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to 0} f(1/x)$

(xii) $\lim\limits_{x\to 0^+} f(-x) = \lim\limits_{x\to 0^-} f(x)$

Limit in Case of Composite Function

$$\lim_{x\to a} f[g(x)] = f\left(\lim_{x\to a} g(x)\right);$$ provided f is continuous at $x = \lim\limits_{x\to a} g(x)$.

For example :

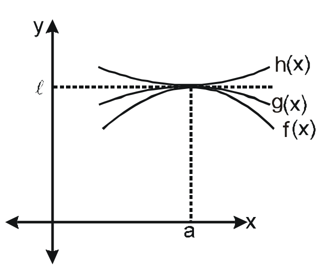
$$\lim_{x\to a} \ell n(f(x)) = \ell n\left\{\lim_{x\to a} f(x)\right\} = \ell n\ \ell$$

$\left(\text{where} \lim\limits_{x\to a} f(x) = \ell,\ \ell > 0\right)$

SANDWITCH THEOREM OR SQUEEZE PLAY THEOREM:

Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in some open interval containing a, except possibly at x = a itself. Suppose also that $\ell im_{x\to a} f(x) = \ell = \ell im_{x\to a} h(x)$,

Then $\ell im_{x\to a} g(x) = \ell$.



Solved Examples

Ex.27 Evaluate $\ell im_{n\to\infty} \dfrac{[x]+[2x]+[3x]+....+[nx]}{n^2}$, where [.] denotes greatest integer function.

Sol. We know that, $x - 1 < [x] \leq x$

$2x - 1 < [2x] \leq 2x$

$3x - 1 < [3x] \leq 3x$

. . .

. . .

$nx - 1 < [nx] \leq nx$

$\therefore$ $(x + 2x + 3x + ... + nx) - n < [x] + [2x] + ... + [nx] \leq (x + 2x + ... + nx)$

$$\Rightarrow \frac{xn(n+1)}{2} - n < \sum_{r=1}^{n}[r\,x] \leq \frac{x.n(n+1)}{2}$$

$$\Rightarrow \ell im_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{1}{n} < \ell im_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} \leq \ell im_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right)$$

$$\Rightarrow \frac{x}{2} < \ell im_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} \leq \frac{x}{2}$$

$$\therefore \ell im_{n\to\infty} \frac{[x]+[2x]+....+[nx]}{n^2} = \frac{x}{2}$$

Aliter

We know that $[x] = x - \{x\}$

$$\sum_{r=1}^{n}[r\,x] = [x] + [2x] + ... + [nx]$$

$= (x + 2x + 3x + ... + nx) - (\{x\} + \{2x\} + ... + \{nx\})$

$$= \frac{xn(n+1)}{2} - (\{x\} + \{2x\} + .. + \{nx\})$$

$$\therefore \frac{1}{n^2}\sum_{r=1}^{n}[r\,x] = \frac{x}{2}\left(1+\frac{1}{n}\right) - \frac{\{x\}+\{2x\}+....+\{nx\}}{n^2}$$

Since, $0 \leq \{rx\} < 1$, $\quad \therefore 0 \leq \sum_{r=1}^{n}\{r\,x\} < n$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{r=1}^{n}\{rx\}}{n^2} = 0$$

$$\therefore \lim_{n\to\infty} \frac{\sum_{r=1}^{n}[rx]}{n^2} = \lim_{n\to\infty} \frac{x}{2}\left(1+\frac{1}{n}\right) - \lim_{n\to\infty} \frac{\sum_{r=1}^{n}\{rx\}}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum_{r=1}^{n}[rx]}{n^2} = \frac{x}{2}$$

METHODS OF REMOVING INDETERMINANCY

Basic methods of removing indeterminancy are

(A) Factorisation

(B) Rationalisation

(C) Using standard limits

(D) Substitution

(E) Expansion of functions.

Factorisation method :-

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaining expression.

Solved Examples

Ex.4 (i) $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-4x+3}$

(ii) $\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$

Sol. (i) $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-4x+3} = \lim_{x\to 3} \frac{(x-3)(x+1)}{(x-3)(x-1)} = 2$

(ii) $\lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x}\right]$

$$= \lim_{x\to 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to 2} \left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

Rationalisation method :-

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Solved Examples

Ex.5 Evaluate :

(i) $\lim_{x\to 1} \frac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$ (ii) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Sol. (i) $\lim_{x\to 1} \frac{4-\sqrt{15x+1}}{2-\sqrt{3x+1}}$

$$= \lim_{x\to 1} \frac{(4-\sqrt{15x+1})(2+\sqrt{3x+1})(4+\sqrt{15x+1})}{(2-\sqrt{3x+1})(4+\sqrt{15x+1})(2+\sqrt{3x+1})}$$

$$= \lim_{x\to 1} \frac{(15-15x)}{(3-3x)} \times \frac{2+\sqrt{3x+1}}{4+\sqrt{15x+1}} = \frac{5}{2}$$

(ii) The form of the given limit is $\frac{0}{0}$ when $x \to 0$.

Rationalising the numerator, we get

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

$$= \lim_{x\to 0} \left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x} \times \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \lim_{x\to 0} \left[\frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}\right]$$

$$= \lim_{x\to 0} \left[\frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}\right]$$

$$= \lim_{x\to 0} \left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{2}{2} = 1$$

Standard limits :

(a) (i) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

[Where x is measured in radians]

(ii) $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}=\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}=1$

(iii) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$; $\lim\limits_{x\to 0}(1+ax)^{\frac{1}{x}}=e^a$

(iv) $\lim\limits_{x\to \infty}\left(1+\dfrac{1}{x}\right)^x=e$; $\lim\limits_{x\to \infty}\left(1+\dfrac{a}{x}\right)^x=e^a$

(v) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$; $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a=\ell n a, a>0$

(vi) $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x}=1$

(vii) $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

(b) If $f(x)\to 0$, when $x\to a$, then

(i) $\lim\limits_{x\to a}\dfrac{\sin f(x)}{f(x)}=1$

(ii) $\lim\limits_{x\to a}\cos f(x)=1$

(iii) $\lim\limits_{x\to a}\dfrac{\tan f(x)}{f(x)}=1$

(iv) $\lim\limits_{x\to a}\dfrac{e^{f(x)}-1}{f(x)}=1$

(v) $\lim\limits_{x\to a}\dfrac{b^{f(x)}-1}{f(x)}=\ell n\, b,\ (b>0)$

(vi) $\lim\limits_{x\to a}\dfrac{\ell n(1+f(x))}{f(x)}=1$

(vii) $\lim\limits_{x\to a}(1+f(x))^{\frac{1}{f(x)}}=e$

(c) $\lim\limits_{x\to a}f(x)=A>0$ and $\lim\limits_{x\to a}\phi(x)=B$ (a finite quantity),

then $\lim\limits_{x\to a}[f(x)]^{\phi(x)}=A^B$.

Solved Examples

Ex.6 Evaluate : $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$

Sol. $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}=\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{(1+x)-1}=n$

Ex.7 Evaluate : $\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{x/2}$

Sol. $\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{x/2}=\lim\limits_{x\to 0}2\times 3\,\dfrac{e^{3x}-1}{3x}=6.$

Ex.8 Evaluate : $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

Sol. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{\tan x(1-\cos x)}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{\tan x\,.\,2\sin^2\frac{x}{2}}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{1}{2}\,.\,\dfrac{\tan x}{x}\,.\left(\dfrac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\dfrac{1}{2}.$

Ex.9 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x}$

Sol. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x}$

$=\lim\limits_{x\to 0}\left[\dfrac{\sin 2x}{2x}\cdot\dfrac{2x}{3x}\cdot\dfrac{3x}{\sin 3x}\right]$

$=\left[\lim\limits_{2x\to 0}\dfrac{\sin 2x}{2x}\right]\cdot\dfrac{2}{3}\cdot\left[\lim\limits_{3x\to 0}\dfrac{3x}{\sin 3x}\right]$

$=1\,.\,\dfrac{2}{3}\times\left[\lim\limits_{3x\to 0}\dfrac{\sin 3x}{3x}\right]$

$=\dfrac{2}{3}\times 1=\dfrac{2}{3}$

Ex.10 Evaluate : $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

Sol. $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^{\lim_{x\to\infty}\frac{2}{x}.x} = e^2.$

Ex.11 Evaluate : (i) $\lim_{x\to3}\frac{e^x-e^3}{x-3}$

(ii) $\lim_{x\to0}\frac{x(e^x-1)}{1-\cos x}$

Sol. (i) Put y = x – 3. So, as $x\to3 \Rightarrow y\to0$. Thus

$$\lim_{x\to3}\frac{e^x-e^3}{x-3} = \lim_{y\to0}\frac{e^{3+y}-e^3}{y}$$

$$= \lim_{y\to0}\frac{e^3.e^y-e^3}{y}$$

$$= e^3\lim_{y\to0}\frac{e^y-1}{y} = e^3 \;.\; 1 = e^3$$

(ii) $\lim_{x\to0}\frac{x(e^x-1)}{1-\cos x} = \lim_{x\to0}\frac{x(e^x-1)}{2\sin^2\frac{x}{2}}$

$$= \frac{1}{2}\,.\,\lim_{x\to0}\left[\frac{e^x-1}{x}\cdot\frac{x^2}{\sin^2\frac{x}{2}}\right] = 2.$$

Use of substitution in solving limit problems

Sometimes in solving limit problem we convert $\lim_{x\to a} f(x)$ into $\lim_{h\to0} f(a+h)$ or $\lim_{h\to0} f(a-h)$ according as need of the problem. (here h is approaching to zero.)

Solved Examples

Ex.12 Evaluate $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{1-\sqrt{2}\sin x}$

Sol. Put $x = \frac{\pi}{4}+h$

$\because\ x\to\frac{\pi}{4} \Rightarrow h\to0$

$$\lim_{h\to0}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{1-\sqrt{2}\sin\left(\frac{\pi}{4}+h\right)} = \lim_{h\to0}\frac{1-\frac{1+\tan h}{1-\tan h}}{1-\sin h-\cos h}$$

$$= \lim_{h\to0}\frac{\frac{-2\tan h}{1-\tan h}}{2\sin^2\frac{h}{2}-2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$= \lim_{h\to0}\frac{-2\tan h}{2\sin\frac{h}{2}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]}\;\frac{1}{(1-\tan h)}$$

$$= \lim_{h\to0}\frac{-2\frac{\tan h}{h}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\left[\sin\frac{h}{2}-\cos\frac{h}{2}\right]}\;\frac{1}{(1-\tan h)} = \frac{-2}{-1} = 2.$$

Limits using expansion

(a) $a^x = 1+\frac{x\,\ell n\,a}{1!}+\frac{x^2\ell n^2 a}{2!}+\frac{x^3\ell n^3 a}{3!}+\,.........,\ a>0$

(b) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......$

(c) $\ell n\,(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....,$ for $-1<x\le1$

(d) $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+.....$

(e) $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+.....$

(f) $\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+......$

(g) $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+....$

(h) $\sin^{-1}x = x+\frac{1^2}{3!}x^3+\frac{1^2.3^2}{5!}x^5+\frac{1^2.3^2.5^2}{7!}x^7+.....$

(i) $\sec^{-1}x = 1+\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}+......$

(j) for $|x| < 1$, $n \in R$; $(1 + x)^n$

$$= 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\infty$$

(k) $(1+x)^{\frac{1}{x}} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \ldots\ldots\ldots\ldots\right)$

Solved Examples

Ex.13 Evaluate $\lim\limits_{x \to 0} \frac{e^x - 1 - x}{x^2}$

Sol. $\lim\limits_{x \to 0} \frac{e^x - 1 - x}{x^2}$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \ldots\ldots\right) - 1 - x}{x^2} = \frac{1}{2}$$

Ex.14 Evaluate $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$

Sol. $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{x^3}$

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3} + \ldots\ldots.\right) - \left(x - \frac{x^3}{3!} + \ldots\ldots\right)}{x^3}$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

Ex.15 Evaluate $\lim\limits_{x \to 1} \frac{(7+x)^{\frac{1}{3}} - 2}{x - 1}$

Sol. Put $x = 1 + h$

$$\lim_{h \to 0} \frac{(8+h)^{\frac{1}{3}} - 2}{h} = \lim_{h \to 0} \frac{2.\left(1 + \frac{h}{8}\right)^{\frac{1}{3}} - 2}{h}$$

$$= \lim_{h \to 0} \frac{2\left\{1 + \frac{1}{3}.\frac{h}{8} + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)\left(\frac{h}{8}\right)^2}{1.2} + \ldots\ldots. - 1\right\}}{h}$$

$$= \lim_{h \to 0} 2 \times \frac{1}{24} = \frac{1}{12}$$

Ex.16 Evaluate $\lim\limits_{x \to 0} \frac{\ell n(1+x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x}$

Sol. $\lim\limits_{x \to 0} \frac{\ell n(1+x) - \sin x + \frac{x^2}{2}}{x \tan x \sin x}$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots..\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots..\right) + \frac{x^2}{2}}{x^3 . \frac{\tan x}{x} . \frac{\sin x}{x}}$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

Ex.17 Evaluate $\lim\limits_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{\tan x}$

Sol. $\lim\limits_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{\tan x} = \lim\limits_{x \to 0} \frac{e - e\left(1 - \frac{x}{2} + \ldots\ldots\right)}{\tan x}$

$$= \lim_{x \to 0} \frac{e}{2} \times \frac{x}{\tan x} = \frac{e}{2}$$

LIMIT WHEN $x \to \infty$

In these types of problems we usually cancel out the greatest power of x common in numerator and denominator both. Also sometime when $x \to \infty$, we use to substitute $y = \frac{1}{x}$ and in this case $y \to 0^+$.

Solved Examples

Ex.18 Evaluate $\lim\limits_{x \to \infty} x \sin \frac{1}{x}$

Sol. $\lim\limits_{x \to \infty} x \sin \frac{1}{x} = \lim\limits_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$

Ex.19 Evaluate $\lim\limits_{x \to \infty} \frac{x - 2}{2x - 3}$

Sol. $\lim\limits_{x \to \infty} \frac{x - 2}{2x - 3} = \lim\limits_{x \to \infty} \frac{1 - \frac{2}{x}}{2 - \frac{3}{x}} = \frac{1}{2}.$

Ex.20 Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2-4x+5}{3x^2-x^3+2}$

Sol. $\lim\limits_{x\to\infty} \dfrac{x^2-4x+5}{3x^2-x^3+2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}-\frac{4}{x^2}+\frac{5}{x^3}}{\frac{3}{x}-1+\frac{2}{x^3}} = 0$

Ex.21 Evaluate $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Sol. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$ (Put $x=-\frac{1}{t}, x\to-\infty \Rightarrow t\to 0^+$)

$$= \lim_{t\to 0^+} \frac{\sqrt{3+2t^2}\cdot\frac{1}{\sqrt{t^2}}}{\frac{(-1-2t)}{t}} = \lim_{t\to 0^+} \frac{\sqrt{3+2t^2}}{-(1+2t)}\frac{t}{|t|}$$

$$= \frac{\sqrt{3}}{-1} = -\sqrt{3}.$$

Some important notes :

(i) $\lim\limits_{x\to\infty} \dfrac{\ell nx}{x} = 0$

(ii) $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$

(iii) $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$

(iv) $\lim\limits_{x\to\infty} \dfrac{(\ell nx)^n}{x} = 0$

(v) $\lim\limits_{x\to 0^+} x(\ell nx)^n = 0$

As $x\to\infty$, $\ell n\, x$ increases much slower than any (positive) power of x where as e^x increases much faster than any (positive) power of x.

(vi) $\lim\limits_{n\to\infty} (1-h)^n = 0$ and $\lim\limits_{n\to\infty} (1+h)^n \to \infty$, where $h\to 0^+$.

Solved Examples

Ex.22 Evaluate $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x}$

Sol. $\lim\limits_{x\to\infty} \dfrac{x^{1000}}{e^x} = 0$

LIMITS OF FORM 1^∞, 0^0, ∞^0

(A) All these forms can be converted into $\frac{0}{0}$ form in the following ways

(a) If $x\to 1, y\to\infty$, then $z=(x)^y$ is of 1^∞ form

$\Rightarrow \ell n\, z = y\, \ell n\, x$

$\Rightarrow \ell n\, z = \dfrac{\ell nx}{\frac{1}{y}}$ $\left(\frac{0}{0}\text{ form}\right)$

As $y\to\infty \Rightarrow \frac{1}{y}\to 0$ and $x\to 1 \Rightarrow \ell nx\to 0$

(b) If $x\to 0, y\to 0$, then $z=x^y$ is of (0^0) form

$\Rightarrow \ell n\, z = y\, \ell n\, x$

$\Rightarrow \ell n\, z = \dfrac{y}{\frac{1}{\ell nx}}$ $\left(\frac{0}{0}\text{ form}\right)$

(c) If $x\to\infty, y\to 0$, then $z=x^y$ is of $(\infty)^0$ form

$\Rightarrow \ell n\, z = y\, \ell n\, x$

$\Rightarrow \ell n\, z = \dfrac{y}{\frac{1}{\ell nx}}$ $\left(\frac{0}{0}\text{ form}\right)$

(B) $(1)^\infty$ type of problems can be solved by the following method

(a) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(b) $\lim\limits_{x\to a} [f(x)]^{g(x)}$; where $f(x)\to 1$;

$g(x)\to\infty$ as $x\to a$

$$= \lim_{x\to a}\,[1+f(x)-1]^{\frac{1}{f(x)-1}\{f(x)-1\}.g(x)}$$

$$= \lim_{x\to a}\left([1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right)^{(f(x)-1)g(x)}$$

$$= e^{\lim_{x\to a}[f(x)-1]\,g(x)}$$

Solved Examples

Ex.23 Evaluate $\lim_{x\to\infty}\left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$

Sol. Since it is in the form of 1^∞

$$\lim_{x\to\infty}\left(\frac{2x^2-1}{2x^2+3}\right)^{4x^2+2}$$

$$= \exp\left(\lim_{x\to\infty}\left(\frac{2x^2-1-2x^2-3}{2x^2+3}\right)(4x^2+2)\right)$$

$$= e^{-8}$$

Ex.24 Evaluate $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$

Sol. Since it is in the form of 1^∞ so $\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$

$$= e^{\lim_{x\to\frac{\pi}{4}}(\tan x-1)\tan 2x} = e^{\lim_{x\to\frac{\pi}{4}}(\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$$

$$= e^{2\times\frac{\tan\pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \frac{1}{e}$$

Ex.25 Evaluate $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$.

Sol. $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ put $x = a + h$

$$= \lim_{h\to 0}\left(1+\frac{h}{(a+h)}\right)^{\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)}$$

$$= \lim_{h\to 0}\left(1+\frac{h}{a+h}\right)^{-\cot\left(\frac{\pi h}{2a}\right)}$$

$$= e^{\lim_{h\to 0}-\cot\frac{\pi h}{2a}.\left(1+\frac{h}{a+h}-1\right)}$$

$$= e^{\lim_{h\to 0}-\left(\frac{\frac{\pi h}{2a}}{\tan\frac{\pi h}{2a}}\right).\frac{\frac{2a}{\pi}}{a+h}} = e^{-\frac{2}{\pi}}$$

Ex.26 Evaluate $\lim_{x\to 0^+} x^x$

Sol. Let $y = \lim_{x\to 0^+} x^x$

$$\Rightarrow \ln y = \lim_{x\to 0^+} x\,\ln x = \lim_{x\to 0^+} -\frac{\ln\frac{1}{x}}{\frac{1}{x}} = 0,$$

as $\frac{1}{x}\to\infty$

$$\Rightarrow\ y = 1$$

'L' HOSPITAL RULE

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the form , $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note :

(1) This rule is applicable only when there is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

(2) This rule is applicable only when the limits of the function exists.

Ex.28 The value of $\lim_{x\to 0}\frac{a^x-b^x}{x}$ is -

(A) log (a/b) (B) log (b/a)

(C) log (ab) (D) 0

Sol. $\lim_{x\to 0}\frac{a^x-b^x}{x}$ $(\frac{0}{0}$ form)

$\Rightarrow \lim_{x\to 0}\frac{a^x\log a-b^x\log b}{1}$ (by 'L'Hospital rule)

$\Rightarrow$ log (a/b)

Ex.29 The value of $\lim_{x\to 0}\frac{x^2-2x}{2\sin x}$ is-

(A) 1 (B) 2

(C) –1 (D) 3

Sol. $\lim_{x\to 0}\frac{x^2-2x}{2\sin x}$ $\left(\frac{\infty}{\infty}\text{form}\right)$

$\Rightarrow \lim_{x\to 0}\frac{2x-2}{2\cos x}=-\frac{2}{2}=-1$ (by L Hospital rule)

Ex.30 $\lim_{x\to 1}\frac{(2x+3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$ equals-

(A) 1/4 (B) 1/2

(C) 3/4 (D) Does not exist

Sol. $\lim_{x\to 1}\frac{(2x+3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$ $\left(\frac{\infty}{\infty}\text{form}\right)$

$\Rightarrow \lim_{x\to 1}\frac{(2)\,(\sqrt{x}-1)+\left(1/2\sqrt{x}\right)(2x+3)}{4x+1}$

(by L Hospital rule)

$\Rightarrow \frac{1/2(5)}{5}=1/2$

Ex.31 If $\lim_{x\to 5}\frac{x^k-5^k}{x-5}=500$, then the positive integral value of k is-

(A) 3 (B) 4

(C) 5 (D) 6

Sol. Here the given Limit

$= k\,5^{k-1}=500$ (given)

$\Rightarrow k.5^{k-1}=4.5^3 \quad \Rightarrow \quad k=4.$

Ex.32 $\lim_{x\to 1} x^{1/1-x}$ equals-

(A) 1 (B) e

(C) 1/e (D) e^2

Sol. $\lim_{x\to 1} x^{1/1-x}$

$=\lim_{x\to 1} e^{(x-1)/1-x}=e^{-1}=1/e$

Ex.33 The value of $\lim_{x\to 64}\left(\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}\right)$ is-

(A) 1 (B) 2

(C) 3 (D) Does not exist

Sol. $\lim_{x\to 64}\left(\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}\right)$ (0/0 form)

$$=\lim_{x\to 64}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{3}x^{-2/3}}=\lim_{x\to 64}\frac{3}{2}\times\frac{x^{2/3}}{x^{1/2}}=\frac{3}{2}\times\frac{16}{8}=3$$

Ex.34 $\lim_{x\to 2}\frac{\sin\left(e^{x-2}-1\right)}{\log\,(x-1)}$ equals-

(A) 0 (B) 1

(C) –1 (D) 2

Sol. $\lim_{x\to 2}\frac{\sin\left(e^{x-2}-1\right)}{\log\,(x-1)}$ (0/0 form)

$$\lim_{x\to 2}\frac{\cos(e^{x-2}-1).e^{x-2}}{\frac{1}{(x-1)}.1}=\frac{\cos 0.e^0}{\frac{1}{2-1}}=1$$

CONITINUITY

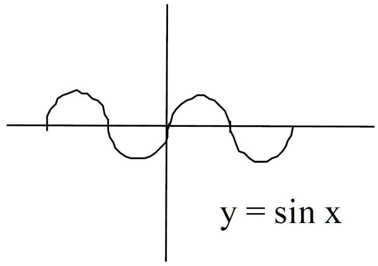
INTRODUCTION

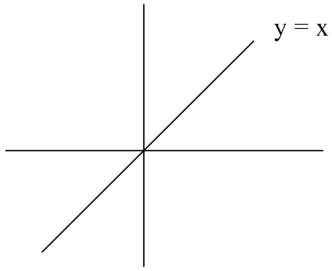
The word **'Continuous'** means without any break or gap. If the graph of a function has no break or gap or jump, then it is said to be **continuous**.

A function which is not continuous is called a **discontinuous function**.

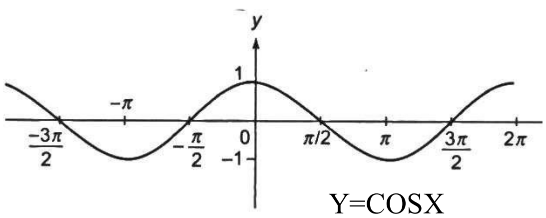
In other words,

If there is slight (finite) change in the value of a function by slightly changing the value of x then function is continuous, otherwise discontinuous, while studying graphs of functions, we see that graphs of functions sin x, x, cos x, e^x etc. are continuous but greatest integer function [x] has break at every integral point, so it is not continuous. Similarly tan x, cot x, secx, 1/x etc. are also discontinuous function.

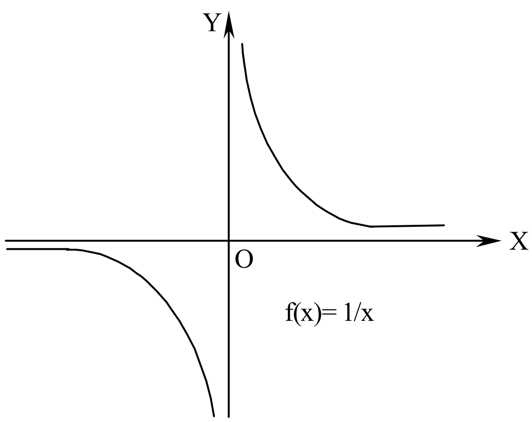




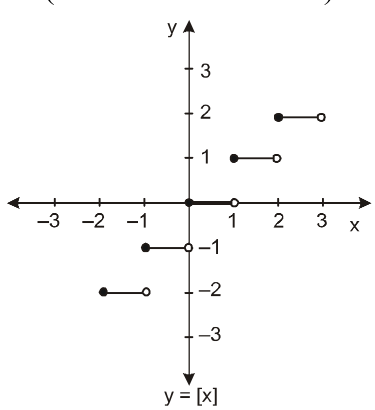
Ex.



(Continuous function)



(Discontinuous at x = 0)



(Discontinuous)

For examining continuity of a function at a point, we find its limit and value at that point, If these two exist and are equal, then function is continuous at that point.

Continuity at a Point

A function f(x) is said to be continuous at x = a, if

$\lim_{x\to a^-} f(a) = \lim_{x\to a^+} f(x) = f(a)$

i.e. LHL = RHL = value of the function at 'a' i.e.

$\lim_{x\to a} f(x) = f(a)$.

If f(x) is not continuous at x = a, we say that f(x) is discontinuous at x = a.

Note :

(i) All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domain .

(ii) We never talk about continuity/discontinuity at a points at which we can't approach from either side of the point. These points are called isolated points e.g. $f(x) = \sqrt{a-x} + \sqrt{x-a}$ at x = a.

(iii) There are some functions which are continuous only at one point.

e.g. $f(x) = \begin{cases} x \text{ if } x\in Q \\ -x \text{ if } x\notin Q \end{cases}$ and $g(x) = \begin{cases} x \text{ if } x\in Q \\ 0 \text{ if } x\notin Q \end{cases}$

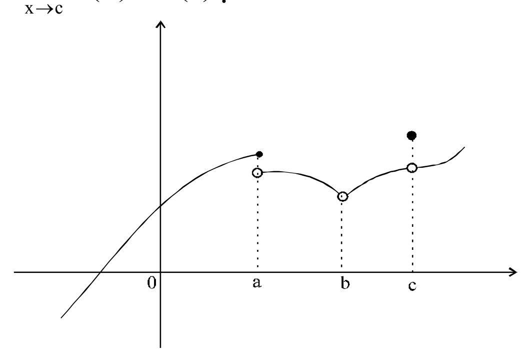
are both continuous only at x = 0

Reasons of Discontinuity

(i) $\lim_{x\to a} f(x)$ does not exist.

(ii) f(x) is not defined at x = b.

(iii) $\lim_{x\to c} f(x) \neq f(c)$.



In all the above cases, geometrically the graph of the function will exhibit a break at the point of discussion. The graph as shown is discontinuous at x = a, b and c.

Solved Examples

Ex.35 Discuss the continuity of the function [cos x] at x $= \frac{\pi}{2}$, where [.] denotes the greatest integer function.

Sol. L.H.L $= \lim_{x\to\frac{\pi}{2}^-}[\cos x] = 0$

R.H.L $= \lim_{x\to\frac{\pi}{2}^+}[\cos x] = -1$

$f\left(\frac{\pi}{2}\right) = \left[\cos\frac{\pi}{2}\right] = 0$

Clearly, L.H.L $\neq$ R.H.L

so, the function is discontinuous at $x = \frac{\pi}{2}$.

Ex.36 If $f(x) = \frac{\sin 2x + A\sin x + B\cos x}{x^3}$ is continuous at x = 0. Find the values of A and B. Also find f(0).

Sol. As f(x) is continuous at x = 0,

$\therefore \lim_{x\to 0} f(x) = f(0)$ and both f(0)

and $\lim_{x\to 0}$ f(x) are finite.

$\Rightarrow f(0) = \lim_{x\to 0}\frac{\sin 2x + A\sin x + B\cos x}{x^3}$

As denominator $\to 0$, when $x \to 0$.

Numerator should also $\to 0$, when $x \to 0$

which is possible only if

$\Rightarrow \sin 2(0) + A\sin(0) + B\cos(0) = 0 \Rightarrow B = 0$

$\therefore f(0) = \lim_{x\to 0}\frac{\sin 2x + A\sin x}{x^3}$

$\Rightarrow f(0) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{2\cos x + A}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{2\cos x + A}{x^2}\right)$

Again we can see that denominator $\to 0$ as $x \to 0$

$\therefore$ Numerator should also approach 0 as $x \to 0$

$\Rightarrow 2 + A = 0 \Rightarrow A = -2$

$\Rightarrow f(0) = \lim_{x\to 0}\left(\frac{2\cos x - 2}{x^2}\right) = \lim_{x\to 0}\left(\frac{-4\sin^2 x/2}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{-\sin^2 x/2}{x^2/4}\right) = -1$

So, we get A = – 2, B = 0 and f(0) = – 1

Ex.37 $f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$ $\forall x \in \left(0, \frac{\pi}{2}\right)$ except at $x = \frac{\pi}{4}$. Define $f\left(\frac{\pi}{4}\right)$ so that f(x) may be continuous at $x = \frac{\pi}{4}$.

Sol. f(x) will be continuous at $x = \frac{\pi}{4}$,

if $\lim_{x\to\pi/4} f(x) = f\left(\frac{\pi}{4}\right)$

$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x\to\pi/4}\frac{\sqrt{2}\cos x - 1}{\cot x - 1}$

$= \lim_{x\to\pi/4}\frac{(\sqrt{2}\cos x - 1)\sin x}{\cos x - \sin x}$

$= \lim_{x\to\pi/4}\frac{(2\cos^2 x - 1)}{(\cos^2 x - \sin^2 x)}\frac{(\cos x + \sin x)\sin x}{(\sqrt{2}\cos x + 1)}$

$= \lim_{x\to\pi/4}\frac{\sin x(\cos x + \sin x)}{\sqrt{2}\cos x + 1} = \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2}.\frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$

Ex.38 Examine the continuity of the function

$f(x) = \begin{cases}\frac{x^2 - 9}{x - 3}, & \text{when } x \neq 3 \\ 6, & \text{when } x = 3\end{cases}$ at x = 3.

Sol. f (3) = 6 (given)

$\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)} = 6$

$\because \lim_{x\to 3} f(x) = f(3)$

$\therefore$ f(x) is continuous at x = 3.

Ex.39 If $f(x) = \begin{cases} \dfrac{\log(1+2ax) - \log(1-bx)}{x} & x \neq 0 \\ k, & x = 0 \end{cases}$

If function is continuous at x = 0 then the value of k is –

(A) a + b (B) 2a + b

(C) a – b (D) 0

Sol. $\lim\limits_{x \to 0} \dfrac{\log\left(\dfrac{1+2ax}{1-bx}\right)}{x}$

$= \lim\limits_{x \to 0} \left(\dfrac{1-bx}{1+2ax}\right) \cdot \dfrac{(1-bx)(2a) - (1+2ax)(-b)}{(1-bx)^2}$

$\Rightarrow \lim\limits_{x \to 0} \dfrac{2a+b}{(1-bx)(1+2ax)} = \dfrac{(2a+b)}{(1)(1)} = 2a + b$

Ex.40 If $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x \neq 0 \\ a, & x = 0 \end{cases}$ is continuous then the value of a is equal to –

(A) 0 (B) 1

(C) 4 (D) 8

Sol. Since the given function is continuous at x = 0

$\lim\limits_{x \to 0} \dfrac{1-\cos 4x}{x^2} = a$

$\lim\limits_{x \to 0} \dfrac{2\sin^2 2x}{x^2} \text{ x } \dfrac{4}{4} = a$

$\lim\limits_{x \to 0} 2\left(\dfrac{\sin 2x}{2x}\right)^2 \text{ x } 4 = a$

$\Rightarrow 2 \text{ x } 1 \text{ x } 4 = a$

$\Rightarrow 8 = a$

Ex.41 Examine the continuity of the function

$f(x) = \begin{cases} x^2 + 1, & \text{when } x \leq 2 \\ 2x, & \text{when } x > 2 \end{cases}$

at the point x = 2.

Sol. $f(2) = 2^2 + 1 = 5$

$f(2-0) = \lim\limits_{h \to 0} \left[(2-h)^2 + 1\right] = 5$

$\because f(2+0) = \lim\limits_{h \to 0} 2(2+h) = 4$

$f(2-0) \neq f(2+0) \neq f(2)$

$\therefore$ f(x) is not continuous at x = 2.

Ex.42 Check the continuity of the function

$f(x) = \begin{cases} x+2 & x > 3 \\ 5 & x = 3 \\ 8-x & x < 3 \end{cases}$ at x = 3.

Sol. f (3) = 5

Left hand limit $\lim\limits_{x \to 3^+} (3+h) + 2$

$\lim\limits_{h \to 0} 5 + h = 5$

Right hand limit $\lim\limits_{x \to 3^-} 8-(3-h)$

$\lim\limits_{h \to 0} 5 + h = 5 = \text{LHL}$

$\because$ f(3) = RHL = LHL

$\therefore$ function is continuous.

Ex.43 If $f(x) = \begin{cases} \dfrac{|x-1|}{1-x} + a & x > 1 \\ a+b & x = 1 \\ \dfrac{|x-1|}{1-x} + b & x < 1 \end{cases}$ is continuous at x = 1 then the value of a & b are respectively-

(A) 1,1 (B) 1,–1

(C) 2,3 (D) None of these

Sol. f(1) = a + b

$f(1+h) = \dfrac{|1+h-1|}{1-(1+h)} + a = -1 + a$

$\because$ given function is continuous

$\therefore$ f(1) = f (1+h)

= a+b = – 1+a $\Rightarrow$ b= –1

Now $f(1-h) = \dfrac{|1-h-1|}{1-(1-h)} + b = \dfrac{h}{h} + b = 1 + b$

$\therefore$ a + b = 1 + b $\Rightarrow$ a = 1

Ex.44 Function f(x) = [x] is a greatest integer function which is right continuous at x = 1 but not left continuous.

Sol. $\because$ f (1) = [1] = 1

[1+0] = 1 and [1-0] =0

$\therefore \lim\limits_{x \to 1^+} f(x) = f(1) = 1,$

and $\lim\limits_{x \to 1^-} f(x) = 0 \neq f(1)$

so function f(x) = [x] is right continuous but not left continuous.

CONTINUITY OF A FUNCTION IN AN INTERVAL

(a) A function f(x) is said to be continuous in an open interval (a,b) if it is continuous at every point in (a, b).

For example function y = sin x, y = cos x , y = e^x are continuous in $(-\infty, \infty)$.

(b) A function f(x) is said to be continuous in the closed interval [a, b] if it is-

(i) Continuous at every point of the open interval (a, b).

(ii) Right continuous at x = a.

(iii) Left continuous at x = b.

Solved Examples

Ex.45 Check the continuity of the function

$$f(x) = \begin{cases} 5x-4, & 0 \le x \le 1 \\ 4x^2-3x, & 1 \le x \le 2 \end{cases} \text{ in an interval } [0,2]$$

Sol. The given function is continuous in the interval [0, 2] because it is right continuous at x = 0 and left continuous at x = 2 and is continuous at every point of the interval (0, 2).

Ex.46 For what value of a and b the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b, & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x, & \pi/2 < x \le \pi \end{cases} \text{ is}$$

continuous in an interval $[0, \pi]$.

Sol. $\because$ f(x) is continuous in an interval $[0, \pi]$

So it is also continuous at x = $\pi/4$,x= $\pi/2$.

$$\therefore \lim_{x \to \pi/4^-} f(x) = \lim_{x \to \pi/4^+} f(x)$$

$$\Rightarrow \pi/4 + a = \pi/2 + b \quad ...(1)$$

$$\text{and} \lim_{x \to \pi/2^-} f(x) = \lim_{x \to \pi/2^+} . f(x)$$

$$\Rightarrow 0+b = -a- b \quad ...(2)$$

Solving (1) and (2) $\Rightarrow$ $a = \pi/6$, $b = -\pi/12$.

Ex.47 Check the continuity of the function f(x)=$[x^2] - [x]^2$ $\forall x \in R$ at the end points of the interval [– 1, 0], where [.] denotes the greatest integer function.

Sol: Continuity at x = – 1

$f(-1) = [(-1)^2] - [-1]^2 = [1] - (-1)^2 = 1 - 1 = 0$

R.H.L = $\lim_{x \to -1^+} \{[x^2] - [x]^2\} = 0 - 1 = -1$

so, $f(-1) \neq$ R.H.L

Continuity at x = 0

$f(0) = [(0)^2] - [0]^2 = 0 - 0 = 0$

LHL = $\lim_{x \to 0^-} \{[x^2] - [x]^2\} = 0 - 1 = -1$

So, $f(0) \neq$ L.H.L

Hence the function is not continuous at the end points of the interval [– 1, 0]

Ex. 48 A function f is defined as follows:

$$f(x) = \begin{cases} 1 & , \text{when } -\infty < x < 0 \\ 1 + \sin x & , \text{when } 0 \le x < \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & , \text{when } \frac{\pi}{2} \le x < \infty \end{cases}$$

Discuss the continuity of f.

Sol. Continuity at x = 0

L.H.L at x = 0 $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1) = 1$

R.H.L at x = 0 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 + \sin x) = 1$

f(0) = 1 + sin 0 = 1

= L.H.L = R.H.L = f(0) so f(x) is continuous at x = 0.

continuity at x = $\frac{\pi}{2}$

L.H.L at x = $\frac{\pi}{2}$ = $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} (1 + \sin x)$

= 1 + 1 = 2

R.H.L at x = $\frac{\pi}{2}$ = $\lim_{x \to \frac{\pi}{2}^+} f(x) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$

$f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$

$\therefore$ L.H.L = R.H.L = $f\left(\frac{\pi}{2}\right)$ $\therefore$ L.H.L=R.H.L= $f\left(\frac{\pi}{2}\right)$

so, f(x) is continuous at x = $\left(\frac{\pi}{2}\right)$

Hence, f(x) is continuous over the whole real number.

CONTINUOUS FUNCTIONS

A function is said to be continuous function if it is continuous at every point in its domain. Following are examples of some continuous function.

(i) $f(x) = x$ (Identity function)

(ii) $f(x) = C$ (Constant function)

(iii) $f(x) = x^2$

(iv) $f(x) = a_0x^n + a_1x^{n-1} + ... + a^n$ (Polynomial).

(v) $f(x) = |x|, x+|x|, x-|x|, x|x|$

(vi) $f(x) = \sin x, f(x) = \cos x$

(vii) $f(x) = e^x, f(x) = a^x, a > 0$

(viii) $f(x) = \log x, f(x) = \log_a x \ a > 0$

(ix) $f(x) = \sinh x, \cosh x, \tanh x$

(x) $f(x) = x^m \sin(1/x), m > 0$

(xi) $f(x) = x^m \cos(1/x), m > 0$

DISCONTINUOUS FUNCTIONS

A function is said to be a discontinuous function if it is discontinuous at at least one point in its domain. Following are examples of some discontinuous function-

(i) $f(x) = 1/x$ at $x = 0$

(ii) $f(x) = e^{1/x}$ at $x = 0$

(iii) $f(x) = \sin 1/x, f(x) = \cos 1/x$ at $x = 0$

(iv) $f(x) = [x]$ at every integer

(v) $f(x) = x - [x]$ at every integer

(vi) $f(x) = \tan x, f(x) = \sec x$

when $x = (2n+1)\ \pi/2, n \in Z$.

(vii) $f(x) = \cot x, f(x) = \text{cosec}\ x$

when $x = n\pi, n \in Z$.

(viii) $f(x) = \coth x, f(x) = \text{cosech}\ x$ at $x = 0$.

PROPERTIES OF CONTINUOUS FUNCTION

The sum, difference, product, quotient (If $Dr \neq 0$) and composite of two continuous functions are always continuous functions. Thus if f(x) and g(x) are continuous functions then following are also continuous functions:

(a) $f(x) + g(x)$

(b) $f(x) - g(x)$

(c) $f(x) . g(x)$

(d) $\lambda\ f(x)$, where λ is a constant

(e) $f(x)/g(x)$, if $g(x) \neq 0$

(f) $f[g(x)]$

For example -

(i) $e^{2x} + \sin x$ is a continuous function because it is the sum of two continuous function e^{2x} and $\sin x$.

(ii) $\sin(x^2+2)$ is a continuous function because it is the composite of two continuous functions $\sin x$ and x^2+2.

Note :

The product of one continuous and one discontinuous function may or may not be continuous.

For example-

(i) $f(x) = x$ is continuous and $g(x) = \cos 1/x$ is discontinuous whereas their product $x \cos 1/x$ is continuous.

(ii) $f(x) = C$ is continuous and $g(x) = \sin 1/x$ is discontinuous whereas their product $C \sin 1/x$ is discontinuous.

Solved Examples

Ex. 49 The function $f(x) = a[x + 1] + b[x - 1]$, where [x] is the greatest integer function then find the condition for which f(x) is continuous at x = 1.

Sol. f(x) is continuous at x = 1

$$\Rightarrow \lim_{x\to 1} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1^-} a[x+1] + b[x-1]$$

$$= \lim_{x\to 1^+} a[x+1] + b[x-1]$$

$$\Rightarrow a - b = 2a + 0b \Rightarrow a + b = 0$$

Continuity of Composite Function

If f is continuous at x = c and g is continuous at x = f(c) then the composite g(f(x)) is continuous at x = c. e.g. $f(x) = \frac{x\sin x}{x^2+2}$ and $g(x) = |x|$ are continuous at x = 0, hence the composite $(gof)(x) = \left|\frac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

Solved Examples

Ex.50 Find the point(s) of discontinuity of $y = \frac{1}{u^2+u-2}$, where $u = \frac{1}{x-1}$.

Sol. The function $u = f(x) = \frac{1}{x-1}$ is discontinuous at the point x = 1. . . . (i)

The function $y = g(x) = \frac{1}{u^2+u-2} = \frac{1}{(u+2)(u-1)}$ is discontinuous at u = – 2 and u = 1.

when $u = -2$, $\frac{1}{x-1} = -2 \quad \Rightarrow x = \frac{1}{2}$

$u = 1 \quad \Rightarrow \frac{1}{x-1} = 1 \quad \Rightarrow x = 2$

Hence, the composite function y = g (f(x)) is discontinuous at three points $x = \frac{1}{2}, 1, 2$

Intermediate Value Theorem

Suppose f(x) is continuous on an interval I, and a and b are two points of I. Then if y_0 is a number between f(a) and f(b), there exists a number c between a and b such that $f(c) = y_0$.

Note :

(a) If f is a continuous function in [a, b] and λ is any real number such that $f(a) < \lambda < f(b)$, then there exists at least one solution of the equation $f(x) = \lambda$ in the open interval (a, b). In general odd number of solution of $f(x) = \lambda$ in the open interval (a, b). In particular if f(a) and f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b). In general odd number of solutions of f(x) = 0 in the open interval (a, b).

(b) If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other I.

In other words if $m = \min_{a\le x\le b} f(x)$, $M = \max_{a\le x\le b} f(x)$, then for any A satisfying the inequalities $m \le A \le M$ there exist a point x_0 Î [a, b] for which $f(x_0) = A$.

(c) A continuous function whose domain is closed must have a range also in closed interval but it is not necessary that domain is open then range is open (range can be closed). f(x) has the minimum and maximum values on [a, b].

DEFINITION OF THE DERIVATIVE

The derivative $f'(x)$ of a function y = f(x) at a given point x is defined as

$$f'(x) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \text{finite.}$$

If this limit exists finitely then the function

f(x) is called differentiable at the point x. The number $f'_-(x)\left(=\lim_{\Delta x\to 0^-}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \text{finite}\right)$ is called the left hand derivative at the point x.

Similarly the number

$$f'_+(x)\left(=\lim_{\Delta x\to 0^+}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \text{finite}\right)$$

is called the right hand derivative at the point x. The necessary and sufficient condition for the existence of the derivative f(x) is the existence of the finite right and left hand derivatives, and also of the equality $f'_+(x) = f'_-(x) = \text{finite}$.

Solved Examples

Ex.51 Show that $\sqrt[5]{x^3}$ has no finite derivative at $x = 0$.

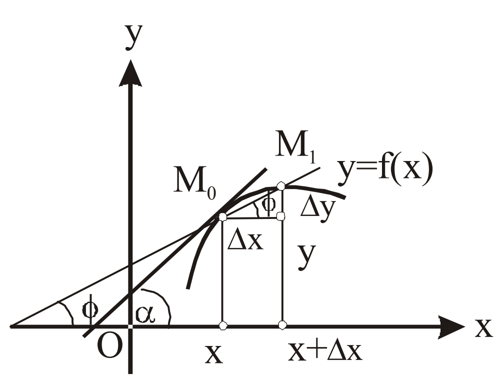
Sol. Let $y = \sqrt[5]{x^3}$, then we have $\Delta y = \sqrt[5]{(x+\Delta x)^3} - \sqrt[5]{x^3}$

At $x = 0$, we have, $\Delta y = \sqrt[5]{\Delta x^3}, \frac{\Delta y}{\Delta x} = \frac{\sqrt[5]{\Delta x^3}}{\Delta x} = \frac{1}{\sqrt[5]{\Delta x^2}}$;

Hence, $y'(0) = \lim\limits_{\Delta x \to 0} \frac{1}{\sqrt[5]{\Delta x^2}} = \infty$ i.e., there is no finite derivative.

Geometrical Meaning of The Derivative

Let us consider the function f(x) and the corresponding curve y = f(x). Clearly line joining two points M_0 (x, y) and M_1 $(x + \Delta x, y + \Delta y)$ on the curve will be the secant to the curve and the slope of this secant is given by $\tan\phi = \frac{\Delta y}{\Delta x}$ (Where ϕ is the angle made by the secant with the positive direction of the x-axis). In the limiting case when $\Delta x \to 0$ the point M_1 approaches M_0 and the secant joining these two points will become the tangent at M_0 whose slope will be given by $\tan\alpha = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$ which means that slope of the tangent to the curve y = f(x) at any argument is equal to the value of the derivative at that argument.



Geometrically, a function is not differentiable in the following cases :

Differentiability on An Interval

A function y = f(x) is differentiable on an interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval [a, b] if it differentiable at every point of the open interval (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \text{finite}$$

(Right - hand derivative at a)

$$\lim_{h \to 0^+} \frac{f(b) - f(b-h)}{h} = \text{finite}$$

(Left - hand derivative at b) exist finitely

Relation Between Derivability And Continuity

(a) If $f'(a)$ exists then $f(x)$ is derivable at $x = a \Rightarrow f(x)$ is continuous at $x = a$ .In general a function f is derivable at x then f is continuous at x. i.e. if f(x) is derivable for every point of its domain of definition, then it is continuous in that domain. The converse of the above result is need not be true e.g.

the functions $f(x) = |x|$ and $g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0, & x = 0 \end{cases}$

both are continuous at x = 0 but not derivable at x = 0.

(b) Let $f'_+(a) = \lambda$ and $f'_-(a) = \mu$ where λ and μ are finite then :

(i) $\lambda = \mu \Rightarrow$ f is derivable at x = a $\Rightarrow$ f is continuous at x = a.

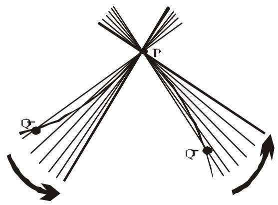
(ii) $\lambda \neq \mu \Rightarrow$ f is not derivable at x = a but f is continuous at x = a. If a function f not differentiable but is continuous at x = a, it geometrically implies a sharp corner or kink at x = a .

(iii) If f is not continuous at x = a then it is not differentiable at x = a.

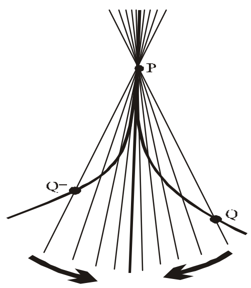
Reason of non differentiability

Case - I

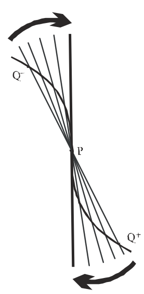
(i) a corner, where the one-sided derivatives differ



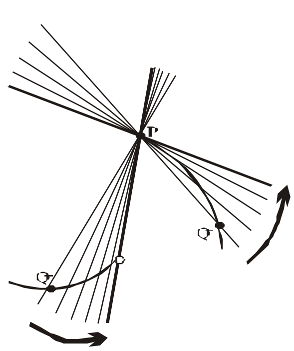
(ii) a cusp, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other



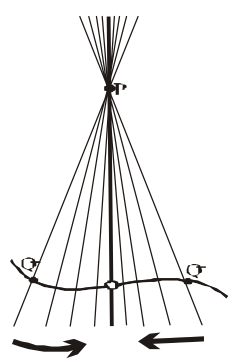
(iii) a vertical tangent, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



(iv) discontinuity



(a)



(b)

Differentiable function & their properties

A function is said to be a differentiable function if it is differentiable at every point of its domain.

(a) Example of some differentiable functions:–

(i) Every polynomial function

(ii) Exponential function : a^x, e^x, e^{-x}......

(iii) logarithmic functions : $\log_a x, \log_e x$,......

(iv) Trigonometrical functions : sin x, cos x,

(v) Hyperbolic functions : sinhx, coshx,......

(b) Examples of some non– differentiable functions:

(i) $|x|$ at x = 0

(ii) $x \pm |x|$ at x = 0

(iii) [x], x $\pm$ [x] at every n $\in$ Z

(iv) $x \sin\left(\frac{1}{x}\right)$, at x = 0

(v) $\cos\left(\frac{1}{x}\right)$, at x = 0

(c) The sum, difference, product, quoteint (Dr $\neq$ 0) and composite of two differentiable functions is always a differentiable function.

Algebra of a Differentiable Function

(i) If f(x) and g(x) are derivable at x = a then the functions f(x) + g(x), f(x) - g(x), f(x).g(x) will be derivable at x = a and if $g(a) \neq 0$ then the function f(x)/g(x) will also be derivable at x = a.

(ii) If f(x) is differentiable at x = a and g(x) is not differentiable at x = a, then the product function f(x) g(x) can still be differentiable at x = a. e.g., f(x) = x and g(x) = |x| at x = 0.

(iii) If f(x) and g(x) both are not differentiable at x = a, then the product function f(x) g(x) can still be differentiable at x = a i.e., f(x) = |x| and g(x) = –|x| at x = 0.

(iv) If f(x) is differentiable at x = a and g(x) is not differentiable at x = a, then the sum function f(x) + g(x) is not - differentiable at x = a.

(v) If f(x) and g(x) both are not - differentiable at x = a, then the sum function may be a differentiable function. e.g. f(x)= |x| + 1 and g(x) = – |x|

(vi) If f(x) is derivable at x = a then it need not be true that $f'(x)$ is continuous at x = a.

e.g. $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Solved Examples

Ex.52 The function f(x) = $x^2 - 2x$ is differentiable at x = 2 because

Sol. $\Rightarrow \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \dfrac{x^2 - 2x - 0}{x - 2}$

$\Rightarrow \lim\limits_{x \to 2} \quad x = 2$

Ex.53 Check the differentiability of the function

$f(x) = \begin{cases} x + 2, & x > 3 \\ 5, & x = 3 \\ 8 - x, & x < 3 \end{cases}$ at x = 3.

Sol. For function to be differentiable

f'(3+h) = f' (3–h)

$f'(3+h) = \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{(3+h+2) - 5}{h} = \lim\limits_{h \to 0} \dfrac{h}{h} = 1$

$f'(3-h) = \lim\limits_{h \to 0} \dfrac{f(3-h) - f(3)}{-h}$

$= \lim\limits_{h \to 0} \dfrac{8 - (3-h) - 5}{-h} = \dfrac{h}{-h} = -1$

$\therefore f'(3+h) \neq f'(3-h)$

So function is not differentiable.

Ex.54 Check the differentiability of the function

$f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at x = 0

Sol. For function to be differentiable

f' (0+h) = f' (0–h)

$f'(0+h) = \dfrac{f(0+h) - f(0)}{h}$

$\Rightarrow \lim\limits_{h \to 0} \dfrac{h \sin\frac{1}{h} - 0}{h} \Rightarrow \lim\limits_{h \to 0} \sin\left(\dfrac{1}{h}\right)$

Which does not exist.

$f'(0-h) = \lim\limits_{h \to 0} \dfrac{(-h)\sin\left(-\frac{1}{h}\right) - 0}{-h}$

$= \lim\limits_{h \to 0} \sin\left(-\dfrac{1}{h}\right)$

Which does not exist.

So function is not differentiable at x = 0

Here we can verify that

f (0+h) = f (0–h) = 0

So function is continuous at x = 0.

Ex.55 Check the differentiability of the function

$f(x) = \begin{cases} 1, & x < 0 \\ 1 + \sin x, & 0 \le x \le \pi/2 \\ 2 + (x - \pi/2)^2, & \pi/2 < x \le \pi \end{cases}$ at $x = \pi/2$

Sol. $f'(\pi/2 + h) = \dfrac{f(\pi/2 + h) - f(\pi/2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2 + (\pi/2 + h - \pi/2)^2 - (1 + \sin \pi/2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2 + h^2 - 1 - 1}{h} = \lim\limits_{h \to 0} h = 0$

$= f'\left(\dfrac{\pi}{2} - h\right) = \dfrac{f\left(\frac{\pi}{2} - h\right) - f\left(\frac{\pi}{2}\right)}{-h}$

$= \lim\limits_{h \to 0} \dfrac{1 + \sin\left(\frac{\pi}{2} - h\right) - \left(1 + \sin\frac{\pi}{2}\right)}{-h}$

$= \lim\limits_{h \to 0} \dfrac{1 + \cos h - 2}{-h} = \lim\limits_{h \to 0} \dfrac{\cos h - 1}{-h} = \lim\limits_{h \to 0} \dfrac{1 - \cosh}{h} = 0$

$\therefore$ function is differentiable at $x = \dfrac{\pi}{2}$

Note : If a function f(x) is discontinuous at x = a then it is not differentiable at that point.