Matrices

INTRODUCTION :

Any rectangular arrangement of numbers (real or complex) (or of real valued or complex valued expressions) is called a **matrix**.This arrangement is enclosed by small ( ) or big [ ] brackets. A matrix is represented by capital letters A, B, C etc. and its element are by small letters a, b, c, x, y etc.

eg. $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, $\qquad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

ORDER OF A MATRIX

A matrix which has m rows and n columns is called a matrix of order m × n.

A matrix A of order m × n is usually written in the following manner-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{23} & a_{23} & ...a_{2j} & ...a_{2n} \\ ..... & ..... & ..... & ..... & ..... \\ a_{i1} & a_{i2} & a_{i3} & ...a_{ij} & ...a_{in} \\ ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & a_{m3} & ...a_{mj} & ...a_{mn} \end{bmatrix} \text{ or}$$

$$A = [a_{ij}]_{m \times n} \quad \text{where} \quad \begin{matrix} i = 1,\ 2,......m \\ j = 1,\ 2,......n \end{matrix}$$

Here a_{ij} denotes the element of i^{th} row and j^{th} column.

eg. order of matrix $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ is 2 × 3.

Solved Examples

Ex.1 Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2} | i - 3j |$.

Sol. In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

$a_{ij} = \frac{1}{2} | i - 3j |$, i = 1, 2, 3 and j = 1, 2

Therefore $a_{11} = \frac{1}{2} | 1 - 3 \times 1 | = 1$

$a_{12} = \frac{1}{2} | 1 - 3 \times 2 | = \frac{5}{2}$

$a_{21} = \frac{1}{2} | 2 - 3 \times 1 | = \frac{1}{2}$

$a_{22} = \frac{1}{2} | 2 - 3 \times 2 | = 2$

$a_{31} = \frac{1}{2} | 3 - 3 \times 1 | = 0$

$a_{32} = \frac{1}{2} | 3 - 3 \times 2 | = \frac{3}{2}$

Hence the required matrix is given hy $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$

TYPES OF MATRICES

ROW MATRIX :

A matrix having only one row is called as row matrix (or row vector).General form of row matrix is $A = [a_{11}, a_{12}, a_{13}, ..., a_{1n}]$

This is a matrix of order "$1 \times n$" (or a row matrix of order n)

COLUMN MATRIX :

A matrix having only one column is called as column matrix (or column vector).

Column matrix is in the form $A = \begin{bmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{bmatrix}$

This is a matrix of order "$m \times 1$" (or a column matrix of order m)

SINGLETON MATRIX :

If in a Matrix there is only one element then it is called Singleton Matrix.

Thus

$A = [a_{ij}]_{m \times n}$ is a Singleton Matrix if m = n = 1.

eg. [2], [3], [a], [–3] are Singleton Matrices.

ZERO OR NULL MATRIX :

$A = [a_{ij}]_{m \times n}$ is called a zero matrix, if $a_{ij} = 0 \; \forall$ i & j.

e.g. : (i) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

SQUARE MATRIX :

A matrix in which number of rows & columns are equal is called a square matrix. The general form of a square matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$$

which we denote as $A = [a_{ij}]_n$.

This is a matrix of order "$n \times n$" (or a square matrix of order n)

Note :

(a) If $m \neq n$ then Matrix is called a Rectangular Matrix.

(b) The elements of a Square Matrix A for which i = j i.e. $a_{11}, a_{22}, a_{33}, ... a_{nn}$ are called diagonal elements and the line joining these elements is called the principal diagonal or of leading diagonal of Matrix A.

(c) **Trance of a Matrix :** The sum of diagonal elements of a square matrix . A is called the trance of Matrix A which is denoted by tr A.

$$\text{tr } A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... a_{nn}$$

DIAGONAL MATRIX :

A square matrix $[a_{ij}]_n$ is said to be a diagonal matrix if $a_{ij} = 0$ for $i \neq j$. (i.e., all the elements of the square matrix other than diagonal elements are zero)

Note :

(a) Diagonal matrix of order n is denoted as Diag $(a_{11}, a_{22}, ...a_{nn})$.

(b) Number of zero in a diagonal matrix is given by $n^2 - n$ where n is a order of the Matrix.

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ = diag (a,b,c)

(ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{bmatrix}$ = diag (a,b,0,c)

SCALAR MATRIX :

Scalar matrix is a diagonal matrix in which all the diagonal elements are same. $A = [a_{ij}]_n$ is a scalar matrix, if (i) $a_{ij} = 0$ for $i \neq j$ and (ii) $a_{ij} = k$ for $i = j$.

e.g. : (i) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

UNIT MATRIX (IDENTITY MATRIX) :

Unit matrix is a diagonal matrix in which all the diagonal elements are unity. Unit matrix of order 'n' is denoted by I_n (or I).

i.e. $A = [a_{ij}]_n$ is a unit matrix when $a_{ij} = 0$ for $i \neq j$ & $a_{ii} = 1$

eg. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

TRIANGULAR MATRIX :

A Square Matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero it is of two types-

UPPER TRIANGULAR MATRIX :

$A = [a_{ij}]_{m \times n}$ is said to be upper triangular, if $a_{ij} = 0$ for $i > j$ (i.e., all the elements below the diagonal elements are zero).

e.g. : (i) $\begin{bmatrix} a & b & c & d \\ 0 & x & y & z \\ 0 & 0 & u & v \end{bmatrix}$ (ii) $\begin{bmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & z \end{bmatrix}$

LOWER TRIANGULAR MATRIX :

$A = [a_{ij}]_{m \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$ for $i < j$. (i.e., all the elements above the diagonal elements are zero.)

e.g. : (i) $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & y & z \end{bmatrix}$ (ii) $\begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ x & y & z & 0 \end{bmatrix}$

Note :

Minimum number of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of Matrix.

COMPARABLE MATRICES :

Two matrices A & B are said to be comparable, if they have the same order (i.e., number of rows of A & B are same and also the number of columns).

e.g. : (i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

are comparable

e.g. : (ii) $C = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -1 & 2 \end{bmatrix}$ & $D = \begin{bmatrix} 3 & 0 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$

are not comparable

EQUALITY OF MATRICES :

Two matrices A and B are said to be equal if they are comparable and all the corresponding elements are equal.

Let $A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{p \times q}$

A = B iff (i) m = p, n = q

(ii) $a_{ij} = b_{ij} \ \forall$ i & j.

SINGULAR MATRIX :

Matrix A is said to be singular matrix if its determinant |A| = 0, otherwise non- singular matrix i.e.

If det | A | = 0 $\Rightarrow$ Singular

and det | A | $\neq 0 \Rightarrow$ non- singular

Solved Examples

Ex.2 : Let $A = \begin{bmatrix} \sin\theta & 1/\sqrt{2} \\ -1/\sqrt{2} & \cos\theta \\ \cos\theta & \tan\theta \end{bmatrix}$ & $B = \begin{bmatrix} 1/\sqrt{2} & \sin\theta \\ \cos\theta & \cos\theta \\ \cos\theta & -1 \end{bmatrix}$.

Find θ so that A = B.

Sol. By definition A & B are equal if they have the same order and all the corresponding elements are equal.

Thus we have $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$ & $\tan\theta = -1$

$\Rightarrow \theta = (2n + 1)\pi - \frac{\pi}{4}$.

Ex.3 If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$,

then find the values of a, b, c, x, y and z.

Sol. As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$x + 3 = 0$ $\quad z + 4 = 6$ $\quad 2y - 7 = 3y - 2$

$a - 1 = -3$ $\quad 0 = 2c + 2$ $\quad b - 3 = 2b + 4$

$\Rightarrow a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$

OPERATIONS ON MATRICES

MULTIPLICATION OF MATRIX BY SCALAR :

Let λ be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product λA is defined as $\lambda A = [b_{ij}]_{m \times n}$ where $b_{ij} = \lambda a_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$ & $-3A \equiv (-3)$

$$A = \begin{bmatrix} -6 & 3 & -9 & -15 \\ 0 & -6 & -3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

Properties of Scalar Multiplication :

If A, B are Matrices of the same order and λ, μ are any two scalars then -

(i) $\lambda(A + B) = \lambda A + \lambda B$

(ii) $(\lambda + \mu) A = \lambda A + \mu A$

(iii) $\lambda(\mu A) = (\lambda \mu A) = \mu(\lambda A)$

(iv) $(-\lambda A) = -(\lambda A) = \lambda(-A)$

(v) tr (kA) = k tr (A)

Note : If A is a scalar matrix, then $A = \lambda I$, where λ is a diagonal entry of A

ADDITION OF MATRICES :

Let A and B be two matrices of same order (i.e. comparable matrices). Then A+B is defined to be.

$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$.

$= [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ $\forall$ i & j.

e.g. : $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ -2 & -3 \\ 5 & 7 \end{bmatrix}$,

$$A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 6 & 7 \end{bmatrix}$$

SUBSTRACTION OF MATRICES :

Let A & B be two matrices of same order. Then A – B is defined as $A + (-B)$ where $-B$ is $(-1)B$.

PROPERTIES OF ADDITION

If A, B and C are Matrices of same order, then-

(i) A + B = B + A (Commutative Law)

(ii) (A+ B) + C = A + (B + C) (Associative Law)

(iii) A + O = O + A = A, where O is zero matrix which is additive identity of the matrix.

(iv) A + (– A) = 0 = (–A) + A where (–A) is obtained by changing the sign of every element of A which is additive inverse of the Matrix

(v) $\left.\begin{matrix} A+B=A+C \\ B+A=C+A \end{matrix}\right\} \Rightarrow$ B = C (Cancellation Law)

(vi) tr (A ± B) = tr (A) ± tr (B)

Solved Examples

Ex.4 If X and Y two matrices are such that $X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ then **Y** matrices is-

(A) $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$ (D) None of these

Sol. Given that $X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ ...(1)

and $X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ ...(2)

Subtracting (2) from (1)

$$-2Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$(-2)Y = \begin{bmatrix} 3-1 & 2-(-2) \\ -1-3 & 0-4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -4 & -4 \end{bmatrix}$$

$$\Rightarrow Y = -\frac{1}{2}\begin{bmatrix} 2 & 4 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$$

Ans.[C]

Ex.5 If A = $\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and A + B – D = 0 (zero matrix) , then D matrix will be-

(A) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 6 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 2 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & -2 \\ -3 & -7 \\ -5 & -6 \end{bmatrix}$

Sol. Let D = $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$\therefore$ A + B – D = $\begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1-1-a & 3-2-b \\ 3+0-c & 2+5-d \\ 2+3-e & 5+1-f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow -a = 0 \Rightarrow a = 0,\ 1 - b = 0$

$\Rightarrow b = 1, 3 - c = 0 \Rightarrow c = 3,\ 7 - d = 0$

$\Rightarrow d = 7,\ 5 - e = 0 \Rightarrow e = 5,\ 6 - f = 0$

$\Rightarrow f = 6$

$\therefore$ D = $\begin{bmatrix} 0 & 1 \\ 3 & 7 \\ 5 & 6 \end{bmatrix}$

Ans.[C]

Ex.6 If $\begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} a & 1 \\ -1 & b \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$,

then value of a, b are -

(A) 1, – 2 (B) – 1, 2

(C) – 1, – 2 (D) 1, 2

Sol. Here

$\begin{bmatrix} 1+a & 0+1 \\ 3-1 & -4+b \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$

$\Rightarrow 1 + a = 2$ and $-4 + b = -2$

$\Rightarrow a = 1, b = 2$

Ans.[D]

Ex.7 If X = $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and 3X – $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

then a is equal to -

(A) 1 (B) 2 (C) 0 (D) – 2

Sol. 3X = $\begin{bmatrix} 3 & 3a \\ 0 & 3 \end{bmatrix}$

$\Rightarrow$ L.H.S. = $\begin{bmatrix} 3-2 & 3a-3 \\ 0-0 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 3a-3 \\ 0 & 1 \end{bmatrix}$

Now by equality of two matrices, we have $3a - 3 = 3 \Rightarrow a = 2$ **Ans.[B]**

Ex.8 IF A = $\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$,

then find the matrix X, such that 2A + 3X = 5B

Sol. We have 2A + 3X = 5B.

$\Rightarrow 3X = 5B - 2A \qquad \Rightarrow X = \frac{1}{3}(5B - 2A)$

$\Rightarrow X = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right)$

$= \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$

$\Rightarrow X = \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -13 & -7 \end{bmatrix}$

$= \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$

Ex.9 If A = $\begin{bmatrix} 1 & -3 & 2 \\ 2 & k & 5 \\ 4 & 2 & 1 \end{bmatrix}$ is a singular matrix, then k is equal to-

(A) – 1 (B) 8 (C) 4 (D) – 8

Sol. A is singular $\Rightarrow |A| = 0$

$\Rightarrow \begin{vmatrix} 1 & -3 & 2 \\ 2 & k & 5 \\ 4 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow 1(k - 10) + 3(2 - 20) + 2(4 - 4k) = 0$

$\Rightarrow 7k + 56 = 0 \Rightarrow k = -8$ **Ans.[D]**

MULTIPLICATION OF MATRICES :

Let A and B be two matrices such that the number of columns of A is same as number of rows of B. i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$, which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}$,

$AB = \begin{bmatrix} 3 & 4 & 9 & 1 \\ 1 & 3 & 7 & 2 \end{bmatrix}$

Notes :

The product AB is defined iff the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined $\not\Rightarrow$ BA is defined.

PROPERTIES OF MATRIX MULTIPLICATION :

If A, B and C are three matrices such that their product is defined , then

(i) $AB \neq BA$ (Generally not commutative)

(ii) $(AB)C = A(BC)$ (Associative Law)

(iii) $IA = A = AI$

I is identity matrix for matrix multiplication

(iv) $A(B + C) = AB + AC$ (Distributive Law)

(v) If $AB = AC$ $\not\Rightarrow$ $B = C$

(Cancellation Law is not applicable)

(vi) If $AB = 0$ It does not mean that $A = 0$ or $B = 0$, again product of two non- zero matrix may be zero matrix.

(vii) $tr(AB) = tr(BA)$

Note :

(i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

(a) $(A + B)^2 = A^2 + B^2 + AB + BA$

(b) $(A - B)^2 = A^2 + B^2 - AB - BA$

(c) $(A - B)(A + B) = A^2 - B^2 + AB - BA$

(d) $(A + B)(A - B) = A^2 - B^2 - AB + BA$

(e) $A(-B) = (-A)B = -(AB)$

Positive Integral powers of a Matrix :

The positive integral powers of a matrix A are defined only when A is a square matrix. Also then

$A^2 = A.A$ $\quad A^3 = A.A.A = A^2A$

Also for any positive integers m,n

(i) $A^m A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn} = (A^n)^m$

(iii) $I^n = I$, $I^m = I$

(iv) $A^o = I_n$ where A is a square matrices of order n.

MATRIX POLYNOMIAL :

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots\dots + a_nx^0$ then we define a matrix polynomial

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots\dots + a_nI_n$

where A is the given square matrix. If f(A) is the null matrix then A is called the zero or root of the matrix polynomial f(x).

Note that $(A)^0$ is not defined if A is a null matrix.

DEFINITIONS :

(A) Idempotent Matrix :

A square matrix is idempotent provide $A^2 = A$. For an idempotent matrix

$A, A^n = A \ \forall \ n \geq 2, n \in N \Rightarrow A^n = A, n \geq 2.$

For example if $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ then $A^2 = A$

i.e. A is idempotent.

B. Nilpotent Matrix:

A square matrix is said to be nilpotent matrix of index p, ($p \in N$), if $A^P = O$, $A^{P-1} \neq 0$ i.e. if p is the least positive integer for which $A^P = O$, then A is said to be nilpotent of index p.

e.g. (i) $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & 3 \end{bmatrix}$ Note that $A^3 = 0$ but $A^2 \neq 0 \Rightarrow$ index of nil potency = 3

(ii) $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is a nilpotent matrix 2.

(iii) $A = \begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix}\begin{bmatrix} a & -a^2 \\ 1 & -a \end{bmatrix}$ nilpoptent

(C) Periodic Matrix :

A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K then A is periodic with period K i.e. if K is the least positive integer for which $A^{K+1} = A$ then A is said to be periodic with period K. If K = 1 then A is called idempotent.

e.g. the matrix $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ has the period 1.

Note:

(1) Period of a square null matrix is not defined.

(2) Period of an idempotent matrix is 1.

(D) Involutory Matrix :

If $A^2 = I$, the matrix is said to be an involutary matrix. An involutary matrix is its own inverse.

e.g. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solved Examples

Ex.10 If $[1 \quad x \quad 2]\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix} = 0$,

then the value of x is -

(A) – 1 (B) 0 (C) 1 (D) 2

Sol. The LHS of the equation

$= [2 \quad 4x + 9 \quad 2x + 5]\begin{bmatrix} x \\ 1 \\ -1 \end{bmatrix}$

$= [2x + 4x + 9 - 2x - 5] = 4x + 4$

Thus $4x + 4 = 0 \Rightarrow x = -1$

Ans.[A]

Ex.11 If $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ then -

(A) x = 2, y = 1 (B) x = 1, y = 2

(C) x = 3, y = 2 (D) x = 2, y = 3

Sol. The given matrix equation can be written as

$\begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\Rightarrow x + 2y = 5$ and $2x + y = 4$

$\Rightarrow x = 1, y = 2$

Ans.[B]

Ex.12 If $A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$ then element a_{21} of A^2 is-

(A) 22 (B) – 15

(C) – 10 (D) 7

Sol. The element a_{21} is product of second row of A to the first column of A

$\therefore a_{21} = [3 \; -4]\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -3 - 12 = -15$

Ans.[B]

TRANSPOSE OF A MATRIX :

Let $A = [a_{ij}]_{m \times n}$. Then the transpose of A is denoted by A'(or A^T) and is defined as

$A' = [b_{ij}]_{n \times m}$ where $b_{ij} = a_{ji} \quad \forall$ i & j.

i.e. A' is obtained by rewriting all the rows of A as columns (or by rewriting all the columns of A as rows).

e.g. : $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ x & y & z & w \end{bmatrix}$, $A' = \begin{bmatrix} 1 & a & x \\ 2 & b & y \\ 3 & c & z \\ 4 & d & w \end{bmatrix}$

Properties of Transpose :

(i) $\left(A^T\right)^T = A$

(ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(AB)^T = B^T A^T$

(iv) $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3 ......A_{n-1}A_n)^T$
$= A_n^T A_{n-1}^T.....A_3^TA_2^TA_1^T$

(vi) $I^T = I$

(vii) tr (A) = tr (A^T)

SYMMETRIC & SKEW-SYMMETRIC MATRIX :

A square matrix A is said to be symmetric if $A' = A$

i.e. Let $A = [a_{ij}]_n$. A is symmetric iff $a_{ij} = a_{ji}$ $\forall$ i & j.

A square matrix A is said to be skew-symmetric if $A' = -A$

i.e. Let $A = [a_{ij}]_n$. A is skew-symmetric iff $a_{ij} = -a_{ji}$ $\forall$ i & j.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

$B = \begin{bmatrix} o & x & y \\ -x & o & z \\ -y & -z & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Properties of Symmetric and skew-symmetric matrices :

(i) If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices while $A - A^T$ is Skew- Symmetric Matrices.

(ii) If A is a Symmetric Matrix, then $-A$, KA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices where $n \in N$, $K \in R$ and B is a square matrix of order that of A.

(iii) If A is a skew symmetric matrix, then-

(a) A^{2n} is a symmetric matrix for $n \in N$

(b) A^{2n+1} is a skew - symmetric matrices for $n \in N$.

(c) kA is also skew - symmetric matrix where $k \in R$.

(d) $B^T AB$ is also skew – symmetric matrix where B is a square matrix of order that of A

(iv) If A, B are two symmetric matrices, then-

(a) $A \pm B$, $AB + BA$ are also symmetric matrices.

(b) $AB - BA$ is a skew - symmetric matrix.

(c) AB is a symmetric matrix when $AB = BA$.

(v) If A,B are two skew - symmetric matrices, then-

(a) $A \pm B$, $AB - BA$ are skew- symmetric matrices.

(b) $AB + BA$ is a symmetric matrix.

(vi) If A is a skew - symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix A can uniquelly be expressed as sum of a symmetric and skew symmetric matrix i.e.

$$A = \left[\frac{1}{2}\left(A + A^T\right)\right] + \left[\frac{1}{2}\left(A - A^T\right)\right]$$

ORTHOGONAL MATRIX

A square Matrix A is called orthogonal if $AA^T = I = A^T A$ i.e. if $A^{-1} = A^T$

eg. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a orthogonal matrix because here $A^{-1} = A^T$

Solved Examples

Ex.13 If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [13 - 6]$, verify that $(AB)' = B'A'$.

Sol. We have $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \ \ 3 \ \ -6]$

Then $AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} [1 \ \ 3 \ \ -6] = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$

Now $A' = [-2 \ \ 4 \ \ 5]$, $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$

$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \ \ 4 \ \ 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$

Clearly $(AB)' = B'A'$

Ex.14 Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Sol. Here $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$

Let $P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix}$

$= \begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}$

Now $P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, Let $Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

Now $Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ -\frac{1}{2} & 0 & -3 \\ -\frac{5}{2} & 3 & 0 \end{bmatrix} = -Q$

Thus $Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now $P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$

Thus, B is reresented as the sum of a symmetric and a skew symmetric matrix.

Ex.15 Show that BAB' is symmetric or skew-symmetric according as A is symmetric or skew-symmetric (where B is any square matrix whose order is same as that of A).

Sol. Case - I A is symmetric $\Rightarrow A' = A$

$(BAB')' = (B')'A'B' = BAB'$

$\Rightarrow$ BAB' is symmetric.

Case - II A is skew-symmetric $\Rightarrow A' = -A$

$(BAB')' = (B')'A'B' = B(-A)B'$

$= -(BAB') \Rightarrow BAB'$ is skew-symmetric

ADJOINT OF A MATRIX

If every element of a square matrix A be replaced by its cofactor in |A|, then the transpose of the matrix so obtained is called the adjoint of matrix A and it is denoted by adj A

Thus if $A = [a_{ij}]$ be a square matrix and F^{ij} be the cofactor of a_{ij} in |A|, then

Adj. $A = [F^{ij}]^T$

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots a_{nn} \end{bmatrix}$, then

Adj. $A = \begin{bmatrix} F_{11} & F_{12} & \dots F_{1n} \\ F_{21} & F_{22} & \dots F_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ F_{n1} & F_{n2} & \dots F_{nn} \end{bmatrix}^T$

eg. if $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$ then

adj $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

Properties of adjoint matrix :

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A\,(\text{adj. } A) = |A|\, I_n = (\text{adj } A)\, A$

(Thus A (adj A) is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj }(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj }(\text{ adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj }(A^T) = (\text{adj } A)^T$

(vi) $\text{adj }(AB) = (\text{adj } B)\,(\text{adj } A)$

(vii) $\text{adj }(A^m) = (\text{adj } A)^m,\ m \in N$

(viii) $\text{adj }(kA) = k^{n-1}\,(\text{adj. } A),\ k \in R$

(ix) $\text{adj }(I_n) = I_n$

(x) $\text{adj } 0 = 0$

(xi) A is symmetric $\Rightarrow$ adj A is also symmetric

(xii) A is diagonal $\Rightarrow$ adj A is also diagonal

(xiii) A is triangular $\Rightarrow$ adj A is also triangular

(xiv) A is singular $\Rightarrow$ $|\text{adj } A| = 0$

Solved Examples

Ex.16 If $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, then adj. A is equal to -

(A) $\begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} -14 & 4 & 22 \\ 4 & 22 & -14 \\ 22 & -14 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 14 & 4 & -22 \\ 4 & -22 & -14 \\ -22 & -14 & -4 \end{bmatrix}$ (D) None of these

Sol. $\text{adj. } A = \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}^T$

$= \begin{bmatrix} 14 & -4 & -22 \\ -4 & -22 & 14 \\ -22 & 14 & -4 \end{bmatrix}$

Ans.[A]

Ex.17 For a 3×3 skew-symmetric matrix A, show that adj A is a symmetric matrix.

Sol. $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ $\text{cof} A = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$

$\text{adj } A = (\text{cof } A)' = \begin{bmatrix} c^2 & -bc & ca \\ -bc & b^2 & -ab \\ ca & -ab & a^2 \end{bmatrix}$

which is symmetric.

Ex.18 If $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, then | adj (adj A) | is equal-

(A) 8 (B) 16 (C) 2 (D) 0

Sol. $|A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2$

$\therefore\ |\text{adj }(\text{adj.}A)| = |A|^{(n-1)^2} = |A|^{2^2}$

$[\because \text{Here } n = 3]\ = 2^4 = 16$ **Ans.[B]**

ELEMENTARY ROW TRANSFORMATION OF MATRIX :

The following operations on a matrix are called as elementary row transformations.

(a) Interchanging two rows.

(b) Multiplications of all the elements of row by a nonzero scalar.

(c) Addition of constant multiple of a row to another row.

Note :

Similar to above we have elementary column transformations also.

Remarks : Two matrices A & B are said to be equivalent if one is obtained from other using elementary transformations. We write $A \approx B$.

If A is a square matrix of order m, and if these exists another square matrix of B of the same order m, such that AB = BA = I. then B is called the inverse matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

Note :

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is neccessary that matrices A and B should be square matrices of the same order .
2. If B is the inverse of A, then A is also the inverse of B.

INVERSE OF A MATRIX

If A and B are two matrices such that

$AB = I = BA$

then B is called the inverse of A and it is denoted by A^{-1} , thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

I. FINDING INVERSE USING ELEMENTRY OPERATIONS

(I) USING ROW TRANSFORMATIONS :

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations,

Step I : Write A = IA and

Step II : Apply a sequence of row operation on A = IA till we get, I = BA.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operatdions simultaneously on X and on the first matrix A of the product AB on RHS.

(II) USING COLUMN RANSFORMATIONS:

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary column operations,

Step I : Write A = AI and

Step II : Apply a sequence of column operations on A = AI till we get, I = AB.

The matrix B will be inverse of A.

Note : In order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply these row operatdions simultaneously on X and on the second matrix B of the product AB on RHS.

Method to find A-1 using elementary operations when A is a 2 ? 2 matrix

Step 1 Make a11 = 1 either by operating

then first operate R1 R2 and then (if need be).

Step II Make a21 = o by the operation .

Step III Make a22 = 1 by the operation .

Step IV Make a12 = 0 by the operation .

Solved Examples

Ex.19 Using elementary transformation. Find the inverse of the following matrices .

Sol. $A = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

Then A = IA $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

operate $R_1 \to \frac{1}{4} R_1 = \begin{bmatrix} 1 & 5/4 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix} A$

Form $R_2 \to R_2 - 3R_1$

$= \begin{bmatrix} 1 & 5/4 \\ 0 & 1/4 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ -3/4 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 5/4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ -3 & 1 \end{bmatrix} A$

From $R2 \to 4R_2$ $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} A$

From $R_1 \to R_1 - 5/4 R_2$

Hence $A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

Ex.20 Obtain the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ using elementary operations.

Sol. Write A = IA, i.e., $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ (applying $R_1 \leftrightarrow R_2$)

or $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$

(applying $R_3 \to R_3 - 3R_1$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$

(applying $R_1 \to R_1 - 2R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$

(applying $R_3 \to R_3 + 5R_2$)

or $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$

(applying $R_3 \to \frac{1}{2} R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$

(Applying $R_1 \to R_1 + R_3$)

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$

(Applying $R_2 \to R_2 - 2R_3$)

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

II. INVERSE USING DETERMINANT OF A MATRIX

To find inverse matrix of a given matrix A we use following formula $A^{-1} = \frac{\text{adj } A}{|A|}$

Thus A^{-1} exists $\Leftrightarrow |A| \neq 0$

Note :

(i) Matrix A is called invertible if A^{-1} exists.

(ii) Inverse of a matrix is unique.

Properties of Inverse Matrix :

Let A and B are two invertible matrices of the same order, then

(i) $(A^T)^{-1} = (A^{-1})^T$

(ii) $(AB)^{-1} = B^{-1} A^{-1}$

(iii) $(A^k)^{-1} = (A^{-1})^k$, $k \in N$

(iv) adj (A^{-1}) = (adj $A)^{-1}$

(v) $(A^{-1})^{-1} = A$

(vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

(vii) If $A = \text{diag}(a_1, a_2, \ldots, a_n)$, then $A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$

(viii) A is symmetric matrix $\Rightarrow$ A^{-1} is symmetric matrix.

(ix) A is triangular matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is a triangular matrix.

(x) A is scalar matrix $\Rightarrow$ A^{-1} is scalar matrix.

(xi) A is diagonal matrix $\Rightarrow$ A^{-1} is diagonal matrix.

(xii) $AB = AC \Rightarrow B = C$, iff $|A| \neq 0$.

Solved Examples

Ex.21 Inverse matrix of $\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}$ is -

(A) $-\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ (B) $-\frac{1}{8}\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$

(C) $\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

Sol. Let the given matrix is A, then $|A| = -8$

and adj $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}$ adj $A = -\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

Ans.[A]

Ex.22 If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $M = AB$, then M^{-1} is equal to -

(A) $\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/6 \end{bmatrix}$

(C) $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$ (D) $\begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/6 \end{bmatrix}$

Sol. $M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$|M| = 6$, adj $M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

$\therefore M^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$

Ans.[C]

Ex.23 $\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$ is equal to-

(A) $\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ (B) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(C) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (D) None of these

Sol. $\begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$

$= \frac{1}{\sec^2\theta/2}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$\therefore$ Product $= \frac{1}{\sec^2\theta/2}$

$\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix}$

$= \frac{1}{\sec^2\theta/2}\begin{bmatrix} 1-\tan^2\theta/2 & -2\tan\theta/2 \\ 2\tan\theta/2 & 1-\tan^2\theta/2 \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta/2-\sin^2\theta/2 & -2\sin\theta/2\cos\theta/2 \\ 2\sin\theta/2\cos\theta/2 & \cos^2\theta/2-\sin^2\theta/2 \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Ans.[C]

Ex.24 If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A adj $A = |A|\,I$. Also find A^{-1}

Sol. We have $|A| = 1(16-9) - 3(4-3) + 3(3-4) = 1 \neq 0$

Now $A_{11} = 7, A_{12} = -1, A_{13} = -1, A_{21} = -3, A_{22} = 1, A_{23} = 0, A_{31} = -3, A_{32} = 0, A_{33} = 1$

Therefore adj $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.\ I$

Also $A^{-1} \frac{1}{|A|}$ adj $A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Ex.25 Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using the equation, find A^{-1}.

Sol. We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore $AA - 4A = -I$

or $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Ex.26 For two non-singular matrices A & B, show that adj (AB) = (adj B) (adj A)

Sol. We have $(AB)(\text{adj}(AB)) = |AB|\, I_n = |A|\,|B|\, I_n$

$A^{-1}(AB)(\text{adj}(AB)) = |A|\,|B|\,A^{-1}$

$\Rightarrow B\,\text{adj}(AB) = |B|\,\text{adj}\,A \quad (\because A^{-1} = \frac{1}{|A|}\,\text{adj}\,A)$

$\Rightarrow B^{-1}B\,\text{adj}(AB) = |B|\,B^{-1}\,\text{adj}\,A$

$\Rightarrow \text{adj}(AB) = (\text{adj}B)(\text{adj}\,A)$

SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS

Consider the system of linear non–homogeneous simultaneous equations in three unknowns x, y and z, given by $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$,

Matrix Method of Solution

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

It can be shown that $AX = B$

Then $X = A^{-1}B$

(a) $\Delta \neq 0$, then A^{-1} exists and hence $AX = B$

$\Rightarrow \quad A^{-1}(AX) = A^{-1}B \quad \Rightarrow \quad x = A^{-1}B$

and therefore unique values of x, y and z are obtained.

(b) We have $AX = B \Rightarrow ((\text{adj}A)A)X = (\text{adj}A)B \Rightarrow \Delta X = (\text{adj}A)B$.

If $\Delta = 0$, then $\Delta X = 0_{3\times 1}$, zero matrix of order 3×1. Now if $(\text{adj}A)B = 0$, then the system $AX = B$ has infinitely many solution, else no solution.

Remember

A system of equation is called consistent if it has a least one solution. If the system has no solution, then it is called inconsistent.

Solved Examples

Ex.27 Solve the following equations by matrix method.

$5x + 3y + z = 16$

$2x + y + 3z = 19$

$x + 2y + 4z = 25$

Sol. Let $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$.

Then the matrix equation of the given system of equations becomes, $AX = B$.

Now $|A| = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 5(4-6) - 3(8-3) + 1(4-1) = -22 \neq 0$.

Hence A is non-negative. Therefore the given system of equations will have the unique solution given by $X = A^{-1}B$.

Let C be the matrix whose elements are the solutions of the corresponding elements of A, then

$$C = \begin{bmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{bmatrix}$$

$$\therefore \text{adj}\,A = C' = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{adj}\,A}{|A|} = \frac{1}{-22}\begin{bmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{22} & \frac{10}{22} & -\frac{8}{22} \\ \frac{5}{22} & -\frac{19}{22} & \frac{13}{22} \\ -\frac{3}{22} & \frac{7}{22} & \frac{1}{22} \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} \frac{2}{22} & \frac{10}{22} & -\frac{8}{22} \\ \frac{5}{22} & -\frac{19}{22} & \frac{13}{22} \\ -\frac{3}{22} & \frac{7}{22} & \frac{1}{22} \end{bmatrix}\begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore x = 1,\ y = 2,\ z = 5$