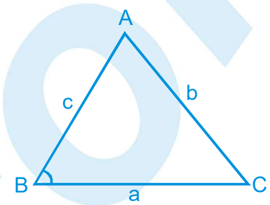
# SOLUTION OF TRIANGLE

## STANDARD SYMBOLS

The process of calculating the sides and angles of triangle using given information is called solution of triangle. In a $\Delta ABC$, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.




Semi-perimeter of the triangle, $s = \dfrac{a+b+c}{2}$

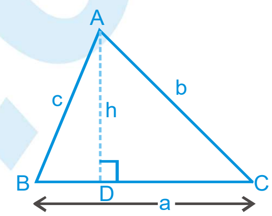
So, $a + b + c = 2s$

The radius of the circumcircle of the triangle, i.e., circumradius = R

The radius of the incircle of the triangle, i.e., inradius = r

Area of the triangle = $\Delta$

## SINE RULE




In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and $\Delta$ is area of triangle.

**Ex.** If $A = 75°$, $B = 45°$, then prove that $b + c\sqrt{2} = 2a$.

**Sol.** $A = 75°$, $B = 45°$ $\Rightarrow$ $C = 60°$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

or $$\frac{a}{\sin 75^\circ} = \frac{b}{\sin 45^\circ} = \frac{c}{\sin 60^\circ} = 2R$$

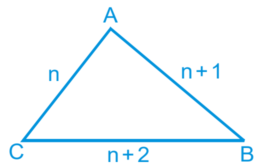
$\Rightarrow$ $$b + c\sqrt{2} = \frac{\sin 45^\circ}{\sin 75^\circ}a + \sqrt{2}\frac{\sin 60^\circ}{\sin 75^\circ}a$$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a + \sqrt{2}\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a = \frac{2}{\sqrt{3}+1}a + \frac{2\sqrt{3}a}{\sqrt{3}+1} = 2a$$

**Ex.** The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

**Sol.** Let the sides be n, n + 1, n + 2 cms.

i.e. AC = n, AB = n + 1, BC = n + 2




Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow$ $3\angle B + \angle C = 180°$ $\Rightarrow$ $\angle C = 180° - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \quad \Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180 - 3B)}{n+1}$$

$$\Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin 3B}{n+1}$$

(i) (ii) (iii)

from **(i)** and **(ii)**;

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \quad \Rightarrow \quad \cos B = \frac{n+2}{2n} \qquad \text{......(iv)}$$

and from **(ii)** and **(iii)**;

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \quad \Rightarrow \quad \frac{\sin B}{n} = \frac{\sin B(3 - 4\sin^2 B)}{n+1}$$

$$\Rightarrow \quad \frac{n+1}{n} = 3 - 4(1 - \cos^2 B) \qquad \text{......(v)}$$

from **(iv)** and **(v)**, we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \quad \Rightarrow \quad \frac{n+1}{n} + 1 = \left(\frac{n^2 + 4n + 4}{n^2}\right)$$

$$\Rightarrow \quad \frac{2n+1}{n} = \frac{n^2 + 4n + 4}{n^2} \quad \Rightarrow \quad 2n^2 + n = n^2 + 4n + 4$$

$$\Rightarrow \quad n^2 - 3n - 4 = 0 \quad \Rightarrow \quad (n-4)(n+1) = 0$$

$n = 4$ or $-1$

where $n \neq -1$

$\therefore$ $n = 4$. Hence the sides are 4, 5, 6

## COSINE RULE

**(I)** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

**(II)** $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ **(III)** $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

**Ex.** In $\Delta ABC$, if $(a + b + c)(a - b + c) = 3ac$, then find $\angle B$.

**Sol.** $(a + c)^2 - b^2 = 3ac$ or $a^2 + c^2 - b^2 = ac$

but $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{1}{2}$ or $B = \dfrac{\pi}{3}$

**Ex.** If $a = \sqrt{3}$, $b = \frac{1}{2}\left(\sqrt{6}+\sqrt{2}\right)$, and $c = \sqrt{2}$, then find $\angle A$.

**Sol.** $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{(1/4)\left(8+4\sqrt{3}\right)+2-3}{\sqrt{12}+\sqrt{4}} = \frac{1+\sqrt{3}}{2(1+\sqrt{3})} = \frac{1}{2}$ or $A = \frac{\pi}{3}$

**Ex.** If in a triangle ABC, $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c} = \frac{a}{bc}+\frac{b}{ac}$, find the $\angle A$.

**Sol.** We have $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c} = \frac{a}{bc}+\frac{b}{ac}$

Multiplying both sides of abc, we get

$\Rightarrow$ $2bc\cos A + ac\cos B + 2ab\cos C = a^2 + b^2$

$\Rightarrow$ $(b^2 + c^2 - a^2) + \frac{(a^2+c^2-b^2)}{2} + (a^2+b^2-c^2) = a^2+b^2$

$\Rightarrow$ $c^2 + a^2 - b^2 = 2a^2 - 2b^2$ $\Rightarrow$ $b^2 + c^2 = a^2$

$\therefore$ $\Delta ABC$ is right angled at A. $\Rightarrow$ $\angle A = 90°$

**Ex.** If in a $\Delta ABC$, $\angle A = 60°$, then find the value of $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)$.

**Sol.** $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right) = \left(\frac{c+a+b}{c}\right)\left(\frac{b+c-a}{b}\right) = \frac{(b+c)^2-a^2}{bc} = \frac{(b^2+c^2-a^2)+2bc}{bc}$

$= \frac{b^2+c^2-a^2}{bc}+2 = 2\left(\frac{b^2+c^2-a^2}{2bc}\right)+2$

$= 2\cos A + 2 = 3$ $\{\because \angle A = 60°\}$

$\therefore$ $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right) = 3$

## PROJECTION FORMULA

In any $\Delta ABC$

**(I)** $a = b\cos C + c\cos B$

**(II)** $b = c\cos A + a\cos C$

**(III)** $c = a\cos B + b\cos A$

**Ex.** In a $\Delta ABC$, $c\cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show a, b, c are in A.P.

**Sol.** Here, $\frac{c}{2}(1+\cos A)+\frac{a}{2}(1+\cos C) = \frac{3b}{2}$

$\Rightarrow$ $a + c + (c\cos A + a\cos C) = 3b$

$\Rightarrow$ $a + c + b = 3b$ {using projection formula}

$\Rightarrow$ $a + c = 2b$

which shows a, b, c are in A.P.

## NAPIER'S ANALOGY (TANGENT RULE)

**(I)** $\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}$ **(II)** $\tan\left(\frac{C-A}{2}\right)=\frac{c-a}{c+a}\cot\frac{B}{2}$

**(III)** $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

**Ex.** Find the unknown elements of the $\Delta ABC$ in which $a=\sqrt{3}+1$, $b=\sqrt{3}-1$, $C=60°$.

**Sol.** $a=\sqrt{3}+1$, $b=\sqrt{3}-1$, $C=60°$

$\Rightarrow$ $A+B+C=180°$

$\therefore$ $A+B=120°$ .......**(i)**

$\Rightarrow$ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$

$=\frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 30°=\frac{2}{2\sqrt{3}}\cot 30°$ $\Rightarrow$ $\tan\left(\frac{A-B}{2}\right)=1$

$\therefore$ $\frac{A-B}{2}=\frac{\pi}{4}=45°$

$\Rightarrow$ $A-B=90°$ .......**(ii)**

From equation **(i)** and **(ii)**, we get

$A=105°$ and $B=15°$

**Ex.** In a triangle ABC, $\angle A=60°$ and $b:c=\sqrt{3}+1:2$, then find the value of $(\angle B-\angle C)$.

**Sol.** $\frac{b}{c}=\frac{\sqrt{3}+1}{2}$ $\Rightarrow$ $\frac{b-c}{b+c}\frac{\sqrt{3}+1-2}{\sqrt{3}+1+2}=\frac{\sqrt{3}-1}{(\sqrt{3}+1)}\frac{1}{\sqrt{3}}$

Now using $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$, we get

$\frac{\sqrt{3}-1}{\left(\sqrt{3}+1\right)}\frac{\sqrt{3}}{\sqrt{3}}=2-\sqrt{3}$ $\Rightarrow$ $\frac{B-C}{2}=15°$

$\therefore$ $B-C=30°$

## TRIGONOMETRIC FUNCTIONS OF HALF ANGLES

**(I)** $\sin\frac{A}{2}=\sqrt{\frac{(s-b)\,(s-c)}{bc}}$, $\sin\frac{B}{2}=\sqrt{\frac{(s-c)\,(s-a)}{ca}}$, $\sin\frac{C}{2}=\sqrt{\frac{(s-a)\,(s-b)}{ab}}$

**(II)** $\cos\frac{A}{2}=\sqrt{\frac{s\,(s-a)}{bc}}$, $\cos\frac{B}{2}=\sqrt{\frac{s\,(s-b)}{ca}}$, $\cos\frac{C}{2}=\sqrt{\frac{s\,(s-c)}{ab}}$

**(III)** $\tan\frac{A}{2}=\sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}=\frac{(s-b)(s-c)}{\Delta}$,

**(IV)** $\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc}$

where $s=\frac{a+b+c}{2}$ is semi perimeter and $\Delta$ is the area of triangle.

**Ex.** If $\cos\frac{A}{2}=\frac{\sqrt{b+c}}{2c}$, then prove that $a^2+b^2=c^2$.

**Sol.** $\cos\frac{A}{2}=\frac{\sqrt{b+c}}{2c}$ $\Rightarrow$ $\frac{s(s-a)}{bc}=\frac{b+c}{2c}$

or $2s(2s-2a)=2b(b+c)$

or $(b+c+a)(b+c-a)=2b^2+2bc$

or $(b+c)^2-a^2=2b^2+2bc$

or $c^2=a^2+b^2$

**Ex.** If $\Delta$ is the area and 2s the sum of the sides of a triangle, then show $\Delta\le\frac{s^2}{3\sqrt{3}}$.

**Sol.** We have, $2s=a+b+c$, $\Delta^2=s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3}\ge[(s-a)(s-b)(s-c)]^{1/3}$$

or $\frac{3s-2s}{3}\ge\left(\frac{\Delta^2}{s}\right)^{1/3}$

or $\frac{s}{3}\ge\left(\frac{\Delta^2}{s}\right)^{1/3}$

or $\frac{\Delta^2}{s}\le\frac{s^3}{27}$ $\Rightarrow$ $\Delta\le\frac{s^2}{3\sqrt{3}}$

## AREA OF TRIANGLE ($\Delta$)

$$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\sqrt{s(s-a)(s-b)(s-c)}$$

**Ex.** Prove that $a^2\sin 2B+b^2\sin 2A=4\Delta$.

**Sol.** $a^2\sin 2B+b^2\sin 2A=4R^2[\sin^2A(2\sin B\cos B)+\sin^2B(2\sin A\cos A)]$

$=8R^2\sin A\sin B(\sin A\cos B+\sin B\cos A)$

$=8R^2\sin A\sin B\sin(A+B)$

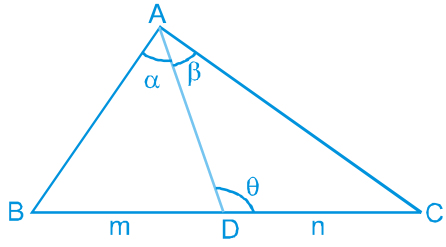
$=8R^2\sin A\sin B\sin C=4\Delta$

## m - n RULE

In any triangle ABC if D be any point on the base BC, such that BD : DC :: m : n and if $\angle BAD=\alpha$, $\angle DAC=\beta$, $\angle CDA=\theta$, then

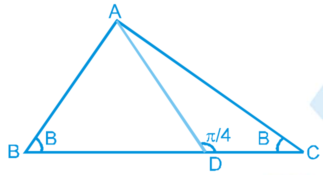
$(m+n)\cot\theta = m\cot\alpha-n\cot\beta$

$= n\cot B-m\cot C$



**Ex.** If the median AD of triangle ABC makes an angle $\pi/4$ with the side BC, then find the value of $|\cot B - \cot C|$.

**Sol.** By m – n theorem




$$(BD + DC)\cot\frac{\pi}{4} = DC\cot B - BD\cot C$$

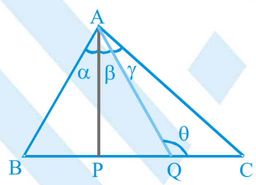
$\Rightarrow$ $|\cot B - \cot C| = 2$

**Ex.** The base of a triangle is divided into three equal parts. If $t_1$, $t_2$, $t_3$ be the tangents of the angles subtended by these parts at the opposite vertex, prove that : $\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right) = 4\left(1+\frac{1}{t_2^2}\right)$

**Sol.** Let the points P and Q divide the side BC in three equal parts :

Such that BP = PQ = QC = x

Also let,




$\angle BAP = \alpha$, $\angle PAQ = \beta$, $\angle QAC = \gamma$

and $\angle AQC = \theta$

From question, $\tan\alpha = t_1$, $\tan\beta = t_2$, $\tan\gamma = t_3$.

Applying

m : n rule in triangle ABC we get,

$$(2x + x)\cot\theta = 2x\cot(\alpha+\beta) - x\cot\gamma \qquad \text{....... (i)}$$

from $\Delta APC$, we get

$$(x + x)\cot\theta = x\cot\beta - x\cot\gamma \qquad \text{....... (ii)}$$

dividing **(i)** and **(ii)**, we get

$$\frac{3}{2} = \frac{2\cot(\alpha+\beta) - \cot\gamma}{\cot\beta - \cot\gamma}$$

or $$3\cot\beta - \cot\gamma = \frac{4(\cot\alpha.\cot\beta - 1)}{\cot\beta + \cot\alpha}$$

or $$3\cot^2\beta - \cot\beta\cot\gamma + 3\cot\alpha.\cot\beta - \cot\alpha.\cot\gamma = 4\cot\alpha.\cot\beta - 4$$

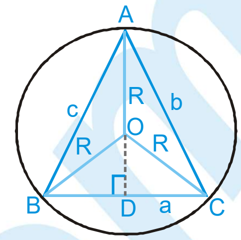
or $$4 + 4\cot^2\beta = \cot^2\beta + \cot\alpha\,.\,\cot\beta + \cot\beta\,.\,\cot\gamma + \cot\gamma\,.\,\cot\alpha$$

or $$4(1 + \cot^2\beta) = (\cot\beta + \cot\alpha)(\cot\beta + \cot\gamma)$$

or $$4\left(1+\frac{1}{t_2^2}\right) = \left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)$$

## RADIUS OF THE CIRCUMCIRCLE 'R'

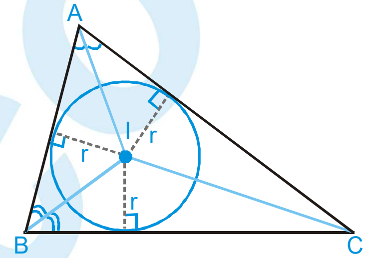
Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.




$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

## RADIUS OF THE INCIRCLE 'r'

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.

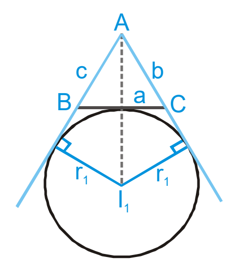



$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

## RADII OF THE EX-CIRCLES

Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If $r_1$ is the radius of described circle opposite to $\angle A$ of $\Delta ABC$ and so on, then -




**(I)** $$r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

**(II)** $$r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$$

**(III)** $$r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

$I_1$, $I_2$ and $I_3$ are taken as ex-centre opposite to vertex A, B, C repsectively.

**Ex.** In a $\Delta ABC$ if a = 13 cm, b = 14 cm and c = 15 cm, then find its circumradius.

**Sol.** $R = \frac{abc}{4\Delta}$ .....**(i)**

✈ $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

✈ $s = \frac{a+b+c}{2} = 21$ cm

$\therefore$ $\Delta = \sqrt{21\times 8\times 7\times 6} = \sqrt{7^2\times 4^2\times 3^2}$ $\Rightarrow$ $\Delta = 84\ cm^2$

$\therefore$ $R = \frac{13\times 14\times 15}{4\times 84} = \frac{65}{8}$ cm $\therefore$ $R = \frac{65}{8}$ cm.

**Ex.** If A, B, C are the angles of a triangle, prove that : $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.

**Sol.** $\cos A + \cos B + \cos C = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right) + \cos C$

$$= 2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right] \quad \left\{\because \frac{C}{2} = 90° - \left(\frac{A+B}{2}\right)\right\}$$

$$= 1 + 2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2} = 1 + 4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$$

$$= 1 + \frac{r}{R} \quad \{\text{as, } r = 4R\sin A/2 . \sin B/2 . \sin C/2\}$$

$\Rightarrow$ $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$. Hence proved.

**Ex.** If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

**Sol.** We have, $r_1 - r = r_2 + r_3$

$\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$ $\quad\Rightarrow \frac{s-s+a}{s(s-a)} = \frac{s-c+s-b}{(s-b)(s-c)}$

$\Rightarrow \frac{a}{s(s-a)} = \frac{2s-(b+c)}{(s-b)(s-c)}$ $\quad\{\text{as, } 2s = a+b+c\}$

$\Rightarrow \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)}$ $\quad\Rightarrow s^2 - (b+c)s + bc = s^2 - as$

$\Rightarrow s(-a+b+c) = bc$ $\quad\Rightarrow \frac{(b+c-a)(a+b+c)}{2} = bc$

$\Rightarrow (b+c)^2 - (a)^2 = 2bc$ $\quad\Rightarrow b^2 + c^2 + 2bc - a^2 = 2bc$

$\Rightarrow b^2 + c^2 = a^2$

$\therefore$ $\angle A = 90°$.

**Ex.** If the area of a $\Delta ABC$ is 96 sq. unit and the radius of the described circles are respectively 8, 12 and 24. Find the perimeter of $\Delta ABC$.

**Sol.** $\Delta = 96$ sq. unit

$r_1 = 8$, $r_2 = 12$ and $r_3 = 24$

✈ $r_1 = \frac{\Delta}{s-a}$ $\Rightarrow$ $s - a = 12$ ......(i)

✈ $r_2 = \frac{\Delta}{s-b}$ $\Rightarrow$ $s - b = 8$ ......(ii)

✈ $r_3 = \frac{\Delta}{s-c}$ $\Rightarrow$ $s - c = 4$ ......(iii)

$\therefore$ adding equations (i), (ii) & (iii), we get

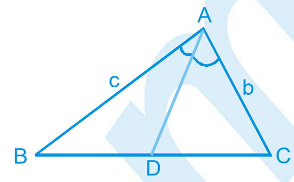
$3s - (a+b+c) = 24$

$s = 24$

$\therefore$ perimeter of $\Delta ABC = 2s = 48$ unit.

## ANGLE BISECTORS, MEDIANS & ALTITUDE

An angle bisector divides the base in the ratio of corresponding sides.




$$\frac{BD}{CD} = \frac{c}{b} \quad \Rightarrow \quad BD = \frac{ac}{b+c} \quad \& \quad CD = \frac{ab}{b+c}$$

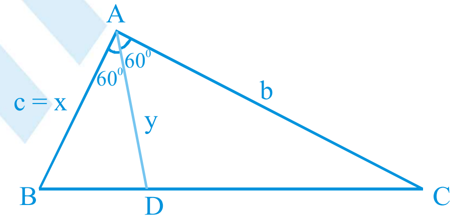
If $m_a$ and $\beta_a$ are the lengths of a median and an angle bisector from the angle A then

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{and} \quad \beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$$

Length of altitude from the angle A $= A_a = \dfrac{2\Delta}{a}$

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

**Ex.** Let ABC be a triangle with $\angle BAC = 2\pi/3$ and AB = x such that (AB) (AC) = 1. If x varies, then find the longest possible length of the angle bisector AD.



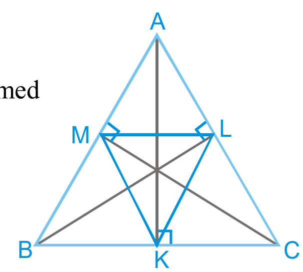

**Sol.** $AD = y = \dfrac{2bc}{b+c}\cos\dfrac{A}{2} = \dfrac{bx}{b+x}$ (as c = x)

But $bx = 1$ or $b = \dfrac{1}{x}$

$\therefore \quad y = \dfrac{x}{1+x^2} = \dfrac{1}{x + \dfrac{1}{x}}$

Thus, $y_{max} = \dfrac{1}{2}$, since the minimum value of the denominator is 2 if x > 0.

## ORTHOCENTRE




**(I)** Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

**(II)** The distances of the orthocentre from the angular points of the $\Delta$ABC are 2R cosA, 2R cosB, & 2R cosC.

**(III)** The distance of P from sides are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB.

**Ex.** If in $\Delta ABC$, the distance of the vertices from the orthocenter are x, y and z, then prove that

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}.$$

**Sol.** We know that distance of orthocenter **(H)** from vertex **(A)** is 2R cos A

or $x = 2R\cos A,\ y = 2R\cos B,\ z = 2R\cos C$

$$\Rightarrow \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{2R\sin A}{2R\cos A}+\frac{2R\sin B}{2R\cos B}+\frac{2R\sin C}{2R\cos C}= \tan A+\tan B+\tan C=\tan A\tan B\tan C$$

Also, $$\frac{abc}{xyz}=\frac{(2R\sin A)(2R\sin B)(2R\sin C)}{(2R\cos A)(2R\cos B)(2R\cos C)}=\tan A+\tan B+\tan C$$

## THE DISTANCES BETWEEN THE SPECIAL POINTS

**(I)** The distance between circumcentre and orthocentre is = $R\sqrt{1-8\cos A\cos B\cos C}$

**(II)** The distance between circumcentre and incentre is $=\sqrt{R^2-2Rr}$

**(III)** The distance between incentre and orthocentre is $=\sqrt{2r^2-4R^2\cos A\cos B\cos C}$

**(IV)** The distances between circumcentre & excentres are

$$OI_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\sqrt{R^2+2Rr_1} \quad \& \quad \text{so on.}$$

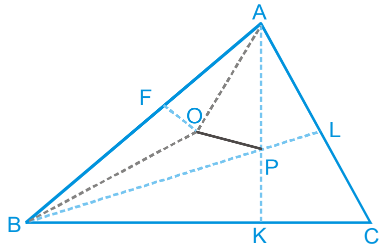
**Ex.** Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1-8\cos A\cos B\cos C}$.

**Sol.** Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have $\angle OAF = 90° - \angle AOF = 90° - C$. Also $\angle PAL = 90° - C$.

Hence, $\angle OAP = A - \angle OAF - \angle PAL = A - 2(90° - C) = A + 2C - 180°$

$= A + 2C - (A + B + C) = C - B$.

Also OA = R and PA = 2RcosA.




Now in $\Delta AOP$,

$OP^2 = OA^2 + PA^2 - 2OA.\ PA\cos OAP$

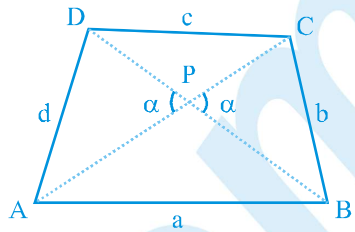
$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)]$

$= R^2 - 4R^2\cos A[\cos(B + C) + \cos(C - B)] = R^2 - 8R^2\cos A\cos B\cos C.$

Hence $OP = R\sqrt{1-8\cos A\cos B\cos C}$.

## AREA OF QUADRILATERAL




Area of $\Delta APD$ + Area of $\Delta DPC$.

Similarly, Area of $\Delta ABC = \frac{1}{2} DP \times AC \times \sin\alpha$

$\therefore$ S = Area of $\Delta DAC$ + Area of $\Delta ABC$

$= \frac{1}{2} DP \times AP \sin\alpha + \frac{1}{2} BP \times AC \sin\alpha$

$= \frac{1}{2}(DP + BP) AC \sin\alpha \Rightarrow S = \frac{1}{2} BD \times AC \sin\alpha$

$\therefore$ Aera of quadrilateral $= \frac{1}{2}$ (Product of the diagonals) × (Sine of included angle)

## SOLUTION OF TRIANGLES (AMBIGUOUS CASES)

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

- If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

  or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ .B and C can be obtained in the similar way.

- If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$. Also

  $\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated. The third side is given by $a = b\frac{\sin A}{\sin B}$

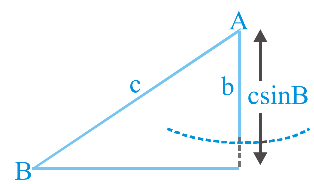
  or $a^2 = b^2 + c^2 - 2bc\cos A$.
- If two sides b and c and an angle opposite the one of them (say B) are given then

  $\sin C = \frac{c}{b}\sin B$, $A = 180° - (B + C)$ and $a = \frac{b\sin A}{\sin B}$ given the remaining elements.

### Case I

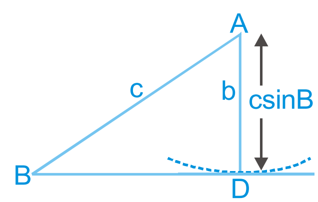
$b < c \sin B$.

We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.



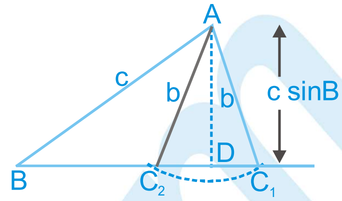

### Case II

$b = c \sin B$ and B is an acute angle, there is only one triangle possible. and it is right-angled at C.



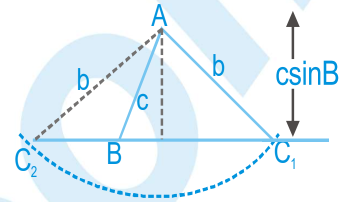
## Case III

b > c sin B, b < c and B is an acute angle, then there are two triangles possible for two values of angle C.



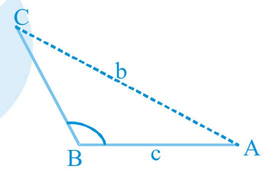

## Case IV

b > c sin B, c < b and B is an acute angle, then there is only one triangle.



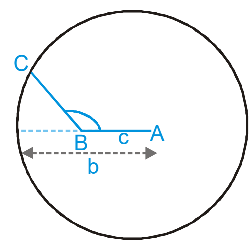

## Case V

b > c sin B, c > b and B is an obtuse angle.
For any choice of point C, b will be greater than c which is a contradiction as c > b (given). So there is no triangle possible.



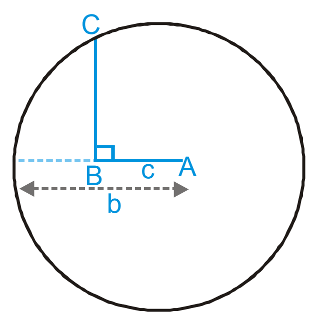

## Case VI

b > c sin B, c < b and B is an obtuse angle.
We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



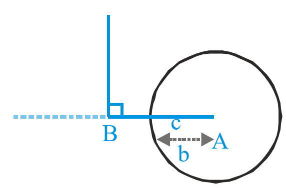

## Case VII

b > c and B = 90°.
Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.




## Case VIII

b ≤ c and B = 90°.
The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.




This is, sometimes, called an ambiguous case.

## Alternative Method

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow$ $a^2 - (2c\cos B)a + (c^2 - b^2) = 0$ $\Rightarrow$ $a = c\cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$

$\Rightarrow$ $a = c\cos B \pm \sqrt{b^2 - (c\sin B)^2}$

This equation leads to following cases :

**Case-I** If $b < c\sin B$, no such triangle is possible.

**Case-II** Let $b = c\sin B$. There are further following case :

**(a)** B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

**(b)** B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

**Case-III** Let $b > c\sin B$. There are further following cases :

**(a)** B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if

$c\cos B > \sqrt{b^2 - (c\sin B)^2}$ or $c > b \Rightarrow$ Two such triangle is possible.

If $c < b$, only one such triangle is possible.

**(b)** B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if $\sqrt{b^2 - (c\sin B)^2} > |c\cos B| \Rightarrow b > c$. So in this case only one such triangle is possible.

If $b < c$ there exists no such triangle.

This is called an ambiguous case.

❖ If one side a and angles B and C are given, then $A = 180° - (B + C)$, and $b = \dfrac{a\sin B}{\sin A}, c = \dfrac{a\sin C}{\sin A}$.

❖ If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

**Ex.** If $b = 3$, $c = 4$ and $B = \pi/3$, then find the number of triangle that can be constructed.

**Sol.** We have,

$$\frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin(\pi/3)}{3} = \frac{\sin C}{4} \quad \text{or} \quad \sin C = \frac{2}{\sqrt{3}} > 1,$$ which is not possible.

Hence, no triangle is possible.

**Ex.** If $A = 30°$, $a = 7$ and $b = 8$ in $\Delta ABC$, then find the number of triangles that can be constructed.

**Sol.** We have, $\frac{a}{\sin A} = \frac{b}{\sin B}$ or $\sin B = \frac{b \sin A}{a} = \frac{8 \sin 30^\circ}{7} = \frac{4}{7}$

Thus, we have, $b > a > b \sin A$.

Hence, angle B has two values given by $\sin B = 4/7$.

**Ex.** If a,b and A are given in a triangle and $c_1, c_2$ are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2\cos^2 A$.

**Sol.** $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow$ $c^2 - 2bc \cos A + b^2 - a^2 = 0$.

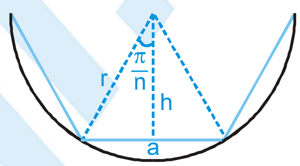
$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$.

$\Rightarrow$ $c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

$= 4b^2 \cos^2 A - 2(b^2 - a^2)2 \cos^2 A = 4a^2\cos^2 A$.

## REGULAR POLYGON

A regular polygon has all its sides equal. It may be inscribed or circumscribed.

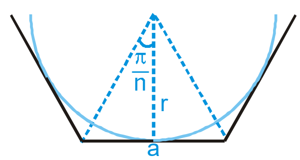



**(a) Inscribed in circle of radius r**

**(i)** $a = 2h \tan\frac{\pi}{n} = 2r \sin\frac{\pi}{n}$

**(ii)** Perimeter **(P)** and area **(A)** of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr \sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2 \sin\frac{2\pi}{n}$

**(b) Circumscribed about a circle of radius r**




**(i)** $a = 2r \tan\frac{\pi}{n}$

**(ii)** Perimeter **(P)** and area **(A)** of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr \tan\frac{\pi}{n}$ and $A = nr^2 \tan\frac{\pi}{n}$

**Ex.** Find the sum of the radii of the circle, which are, respectively, inscribed and circumscribed about a polygon of n sides, whose side length is a.

**Sol.** Radius of the circumscribed circle $= R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$

and Radius of the inscribed circle $= r = \frac{1}{2} a \cot\left(\frac{\pi}{n}\right)$

$\Rightarrow$ $R + r = \frac{a}{2\sin(\pi/n)} + \frac{a\cos(\pi/n)}{2\sin(\pi/n)} = \frac{a[1+\cos(\pi/n)]}{2 \times 2\sin(\pi/2n)\cos(\pi/2n)} = \frac{1}{2}a\cot\left(\frac{\pi}{2n}\right)$

## IMPORTANT POINTS

**(a)** **(i)** If a cos B = b cos A, then the triangle is isosceles.

**(ii)** If a cos A = b cos B, then the triangle is isosceles or right angled.

**(b)** In right angle triangle

**(i)** $a^2 + b^2 + c^2 = 8R^2$ **(ii)** $\cos^2 A + \cos^2 B + \cos^2 C = 1$

**(c)** In equilateral triangle

**(i)** $R = 2r$ **(ii)** $r_1 = r_2 = r_3 = \frac{3R}{2}$

**(iii)** $r : R : r_1 = 1 : 2 : 3$ **(iv)** area $= \frac{\sqrt{3}a^2}{4}$ **(v)** $R = \frac{a}{\sqrt{3}}$

**(d)** **(i)** The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

**(ii)** The orthocentre of right angled triangle is the vertex at the right angle.

**(iii)** The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle. In equilateral triangle, all these centres coincide

**(e)** Area of a cyclic quadrilateral $= \sqrt{s(s-a)(s-b)(s-c)(s-d)}$

where a, b, c, d are lengths of the sides of quadrilateral and $s = \frac{a+b+c+d}{2}$

**Ex.** For a $\Delta ABC$, it is given that $\cos A + \cos B + \cos C = 3/2$. Prove that the triangle is equilateral.

**Sol.** If a, b, c are the sides of the $\Delta ABC$, then $\cos A + \cos B + \cos C = 3/2$

$$\Rightarrow \frac{b^2+c^2-a^2}{2bc} + \frac{a^2+c^2-b^2}{2ac} + \frac{a^2+b^2-c^2}{2ab} = \frac{3}{2}$$

$$\Rightarrow ab^2 + ac^2 - a^3 + bc^2 + ba^2 - b^3 + ca^2 + cb^2 - c^3 = 3abc$$

$$\Rightarrow ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$$

$$\Rightarrow a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = \frac{(a+b+c)}{2}\left\{(a-b)^2 + (b-c)^2 + (c-a)^2\right\}$$

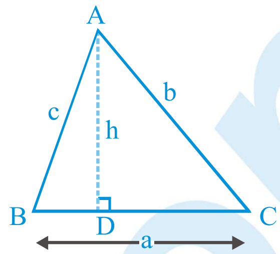
$$\Rightarrow (a+b-c)(a-b)^2 + (b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 = 0 \quad \text{..........} \textbf{(i)}$$

as we know $a + b > c$, $b + c > a$, $c + a > b$

$\therefore$ each term on the left side of equation **(i)** has positive coefficient multiplied by perfect square, each must be separately zero.

$\Rightarrow$ $a = b = c$.

Hence $\Delta$ is equilateral.




## 1. Sine Formulae

In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\lambda=\frac{abc}{2\Delta}=2R$$

where R is circumradius and $\Delta$ is area of triangle

## 2. Cosine Formulae

(a) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ or $a^2=b^2+c^2-2bc\cos A$

(b) $\cos B=\dfrac{c^2+a^2-b^2}{2ca}$

(c) $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

## 3. Projection Formulae

(a) $b\cos C+c\cos B=a$

(b) $c\cos A+a\cos C=b$

(c) $a\cos B+b\cos A=c$

## 4. Napier's Analogy (Tangent Rule)

(a) $\tan\left(\dfrac{B-C}{2}\right)=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(b) $\tan\left(\dfrac{C-A}{2}\right)=\dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(c) $\tan\left(\dfrac{A-B}{2}\right)=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

## 5. Half Angle formulae

(a) (i) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

(ii) $\sin\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{ca}}$

(iii) $\sin\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$

(ii) $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ca}}$

(iii) $\cos\dfrac{C}{2}=\sqrt{\dfrac{s(s-c)}{ab}}$

(c) (i) $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}=\dfrac{\Delta}{s(s-a)}$

(ii) $\tan\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}=\dfrac{\Delta}{s(s-b)}$

(iii) $\tan\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}=\dfrac{\Delta}{s(s-c)}$

(d) **Area of Triangle**

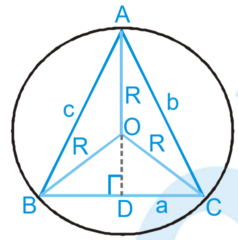
$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$$

$$=\frac{1}{4}\sqrt{2\left(a^2b^2+b^2c^2+c^2a^2\right)-a^4-b^4-c^4}$$

## 6. Radius of The Circumcircle 'R'

Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumcentre 'R'.

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$



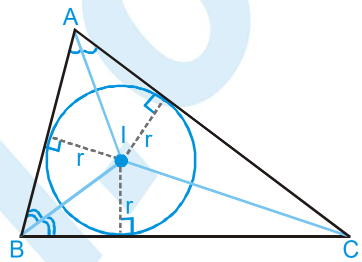

## 7. Radius of The Incircle 'r'

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$$

$$= (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

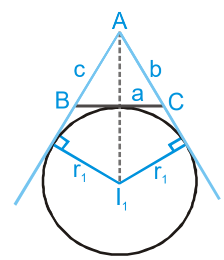
$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$




## 8. Radii of The Ex-Circles

Point of intersection of two external angle and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If $r_1$ is the radius of described circle opposite to angle A of ΔABC and so on then :

**(a)** $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$




**(b)** $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$

**(c)** $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

## 9. Length of Angle Bisector, Median & Altitude

If $m_a$, $\beta_a$ & $h_a$ are the lengths of a median, an angle bisector & altitude from the angle A then,

$$\frac{1}{2}\sqrt{b^2 + c^2 2bc\cos A} = m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

and $\beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}, h_a = \frac{a}{\cot B + \cot c}$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$

## 10. Orthocentre and Pedal Triangle

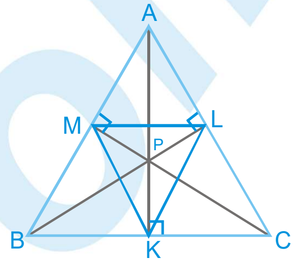
**(a)** Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.

**(b)** The distances of the orthocentre from the angular points of the $\Delta ABC$ are 2R cosA, 2R cosB, & 2R cosC.

**(c)** The distance of orthocentre from sides are 2R cosB cosC, 2RcosC cosAand 2R cosA cosB

**(d)** The sides of the pedal triangle are $a \cos A\ (= R \sin 2A)$, $b\cos B\ (= R \sin 2B)$ and $c \cos C\ (= R \sin 2C)$ and its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$

**(e)** Circumradii of the triangles PBC, PCA, PAB and ABC are equal .




**(f)** Area of pedal triangle = 2D cos A cos B cos C = $\frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C$

**(g)** Circumradii of pedal triangle = R/2

## 11. Ex-Central Triangles

**(a)** The triangle formed by joining the three excentres $I_1$, $I_2$ and $I_3$ of $\Delta ABC$ is called the excentral or excentric triangle.

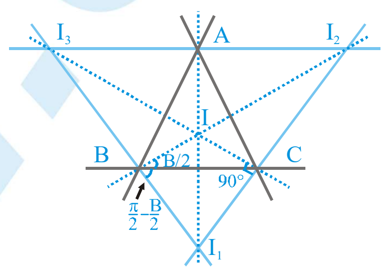
**(b)** Incentre I of $\Delta ABC$ is the orthocentre of the excentral $\Delta\, I_1I_2I_3$.

**(c)** $\Delta ABC$ is the pedal triangle of the $\Delta\, I_1I_2I_3$.

**(d)** The sides of the excentral triangle are

$$4R \cos \frac{A}{2},\ 4R \cos \frac{B}{2} \text{ and } 4R \cos \frac{C}{2}$$

and its angles are $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$.




**(e)** $II_1 = 4R \sin \frac{A}{2}$ ; $II_2 = 4R \sin \frac{B}{2}$ ; $II_3 = 4R \sin \frac{C}{2}$.

## 12. The Distance between the Special Points

**(a)** The distance between circumcentre and orthocentre is $= R\sqrt{1 - 8 \cos A \cos B \cos C}$

**(b)** The distance between circumcentre and incentre is $= \sqrt{R^2 - 2Rr}$

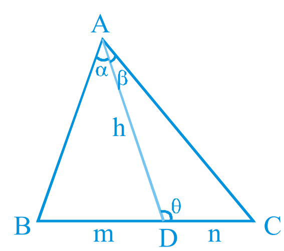
**(c)** The distance between incentre and orthocentre is $= \sqrt{2r^2 - 4R^2 \cos A \cos B \cos C}$

**(d)** The distance between circumcentre & excentre are

$$OI_1 = R\sqrt{1 + 8 \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}} = \sqrt{R^2 + 2Rr_1} \text{ \& so on.}$$

## 13. m-n Theorem

$(m + n) \cot \theta = m \cot \alpha - n \cot \beta$




$(m + n) \cot \theta = n \cot B - m \cot C$.

## 14. Important Points

**(a)** **(i)** If a cos B = b cos A, then the triangle is isosceles.

**(ii)** If a cos A = bcos B, then the triangle is isosceles or right angled.

**(b)** In Right Angle Triangle :

**(i)** $a^2 + b^2 + c^2 = 8R^2$ **(ii)** $\cos^2 A + \cos^2 B + \cos^2 C = 1$

**(c)** In equilateral triangle :

**(i)** $R = 2r$ **(ii)** $r_1 = r_2 = r_3 = \frac{3R}{2}$ **(iii)** $r : R : r_1 = 1 : 2 : 3$

**(iv)** area $= \frac{\sqrt{3}a^2}{4}$ **(v)** $R = \frac{a}{\sqrt{3}}$

**(d)** **(i)** The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

**(ii)** The orthocentre, of right angled triangle is the vertex at the right angle.

**(iii)** The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1,except in case of equilateral triangle. In equilateral triangle all these centres coincide.

## 15. Regular Polygon

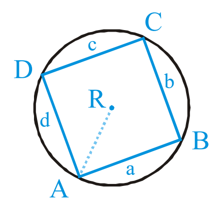
Consider a 'n' sided regular polygon of side length 'a'

**(a)** Radius of incircle of this polygon $r = \frac{a}{2}\cot\frac{\pi}{n}$ **(b)** Radius of circumcircle of this polygon $R = \frac{a}{2}\text{cosec}\frac{\pi}{n}$

**(c)** Perimeter & area of regular polygon Perimeter $= na = 2nr\tan\frac{\pi}{n} = 2nR\sin\frac{\pi}{n}$

$$\text{Area} = \frac{1}{2}nR^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{1}{4}na^2\cot\frac{\pi}{n}$$

## 16. Cyclic Quadrilateral



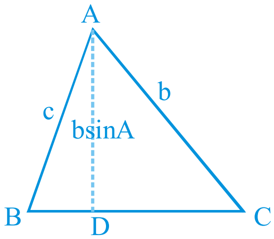

**(a)** Quadrilateral ABCD is cyclic if $\angle A + \angle C = \pi = \angle B + \angle C$ (opposite angle are supplementary angles)

**(b)** Area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$ , where $2s = a + b + c + d$

**(c)** $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$ & similarly other angles

**(d)** Ptolemy`s theorem : If ABCD is cyclic quadrilateral, then AC. BD = AB. CD + BC. AD

## 17. Solution of Triangle




**Case - I :** Three sides are given then to find out three angles use

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ac}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

**Case - II :** Two sides & included angle are given :

Let sides a, b & c are given then use $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$ and find value of $A - B$ ......**(i)**

& $\frac{A+B}{2} = 90^\circ - \frac{C}{2}$ ......**(ii)** $c = \frac{a\sin C}{\sin A}$ ......**(iii)**

**Case - III :** Two sides a, b & angle A opposite to one of them is given

**(a)** If $a < b\sin A$ No triangle exist

**(b)** If $a = b\sin A$ & A is acute, then one triangles exist which is right angled.

**(c)** $a > b\sin A$, $a > b$ & A is acute, then two triangles exist

**(d)** $a > b\sin A$, $a > b$ & A is acute, then there one triangle exist

**(e)** $a > b\sin A$ & A is obtuse, then there is one triangle if $a > b$ & no triangle if $a < b$.