Determinants

DETERMINANT OF A SQUARE MATRIX :

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix.

Let $A = [a]_{1\times 1}$ be a 1×1 matrix. Determinant A is defined as $|A| = a$.

e.g. $A = [-3]_{1\times 1}$ $|A| = -3$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A|$ is defined as ad – bc.

e.g. $A = \begin{bmatrix} 5 & 3 \\ -1 & 4 \end{bmatrix}$, $|A| = 23$

DETERMINANT OF ANY ORDER :

Take any row (or column) ; the value of the determinant is the sum of products of the elements of the row (or column) and the corresponding determinant obtained by omitting the row and the column of the element with a proper sign, given by the rule $(-1)^{r+s}$, where r and s are the number of rows and the number of column respectively of the element of the row (or the column) chosen

The 9 numbers a_r, b_r, c_r (r = 1, 2, 3) arranged as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is determinant of third order.

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

The diagonal through the left-hand top corner which contains the element a_1, b_2, c_3 is called the leading diagonal or principal diagonal and the terms are called the leading terms. The expanded form of determinant has 3! terms

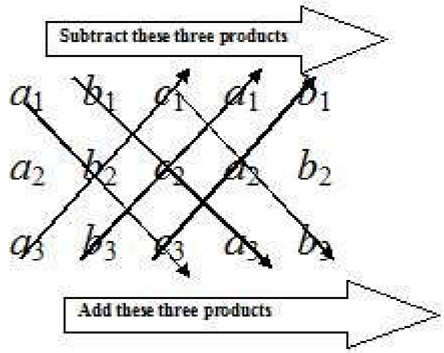
Sarrus Rule (Short cut method):

This method only works for 3×3 matrices. Given a matrix A of order 3×3.

To apply sarrus rule, copy the first and second column of A to form fourth and fifth columns. The determinant of A is then obtained by adding the products of the three "DOWNWARD DIAGONALS" and subtracting the products of the three "UPWARD DIAGONALS" as shown

Thus, the determinant of 3 × 3 matrix A is given by the following

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$



$|A| = (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$

Solved Examples

Ex.1 The value of $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix}$ is -

(A) $2a^2$ (B) 0

(C) – 3 (D) 3

Sol. $\begin{vmatrix} a+1 & a-2 \\ a+2 & a-1 \end{vmatrix}$

$= (a + 1)(a - 1) - (a + 2)(a - 2)$

$= (a^2 - 1) - (a^2 - 4) = 3$ **Ans. [D]**

Ex.2 The value of $\begin{vmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{vmatrix}$ is -

(A) 2 (B) – 1

(C) 0 (D) cos 2θ

Sol. $\begin{vmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{vmatrix}$

$= (1 + \cos\theta)(1 - \cos\theta) - (\sin\theta)(\sin\theta)$

$= 1 - \cos^2\theta - \sin^2\theta = 0$ **Ans. [C]**

Ex.3 The value of $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$ is –

(A) 213 (B) – 231

(C) 231 (D) 39

Sol. $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix} =$

$1\begin{vmatrix} 3 & 6 \\ -7 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 2 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 3 \\ 2 & -7 \end{vmatrix}$

$= 1\,(3 \times 9 - 6\,(-7)) - 2\,(-4 \times 9 - 2 \times 6) + 3\,[(-4)(-7) - 3 \times 2]$

$= (27 + 42) - 2\,(-36 - 12) + 3\,(28 - 6)$

$= 231$ **Ans. [C]**

MINORS & COFACTORS :

Let Δ be a determinant. Then minor of element a_{ij}, denoted by M_{ij}, is defined as the determinant of the submatrix obtained by deleting i^{th} row & j^{th} column of Δ. Cofactor of element a_{ij}, denoted by C_{ij}, is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

e.g.1 $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$M_{11} = d = C_{11}$ $M_{12} = c,\ C_{12} = -c$

$M_{21} = b,\ C_{21} = -b$ $M_{22} = a = C_{22}$

e.g. 2 $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$M_{11} = \begin{vmatrix} q & r \\ y & z \end{vmatrix} = qz - yr = C_{11}$.

$M_{23} = \begin{vmatrix} a & b \\ x & y \end{vmatrix} = ay - bx,\ C_{23} = -(ay - bx)$

$= bx - ay$ etc.

Properties of minor & cofactor :

(i) The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant i.e.

$\Delta = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$

(ii) The sum of the product of element of any row with corresponding cofactor of another row is equal to zero i.e.

$a_{11}F_{21} + a_{12}F_{22} + a_{13}F_{23} = 0$

(iii) If order of a determinant (Δ) is 'n' then the value of the determinant formed by replacing every element by its cofactor is Δ^{n-1}

TRANSPOSE OF A DETERMINANT :

The transpose of a determinant is the determinant of transpose of the corresponding matrix.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

PROPERTIES OF DETERMINANT :

(1) $|A| = |A'|$ for any square matrix A.

i.e. the value of a determinant remains unaltered, if the rows & columns are inter changed,

$$\text{i.e. } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$$

Note :

Since the Determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'Columns'

(2) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

$$\text{e.g. Let } D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ \& \ D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $D_2 = -D_1$

(3) Let λ be a scalar. Than $\lambda\,|A|$ is obtained by multiplying any one row (or any one column) of $|A|$ by λ

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } E = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $E = KD$

(4) $|AB| = |A|\,|B|$.

(5) $|\lambda A| = \lambda^n |A|$, when $A = [a_{ij}]_n$.

(6) A skew-symmetric matrix of odd order has deteminant value zero and of even order is a perfect square.

(7) If a determinant has all the elements zero in any row or column, then its value is zero,

$$\text{i.e. } D = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(8) If a determinant has any two rows (or columns) identical (or proportional), then its value is zero,

$$\text{i.e. } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(9) If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants, i.e.

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(10) The value of a determinant is not altered by adding to the elements of any row (or column) a constant multiple of the corresponding elements of any other row (or column),

$$\text{i.e. } D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D_2$$

$$= \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}. \text{ Then } D_2 = D_1$$

Note :

It should be noted that while applying Property-9 at least one row (or column) must remain unchanged

SOME IMPORTANT DETERMINANTS TO REMEMBER

(1) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)\,(y-z)\,(z-x)$

Proof:

Let $D = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

$R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$

$\Rightarrow D = \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$

$D = (x-y)(y-z) \begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$

$= (x-y)(y-z)\ (z-x)$

$D = (x-y)(y-z)(z-x)$.

Hence Proved.

(2) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

Proof :

Let $D = \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$

Apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ Given

$D = \begin{vmatrix} 0 & x-y & x^3-y^3 \\ 0 & y-z & y^3-z^3 \\ 1 & z & z^3 \end{vmatrix}$

$= (x-y)(y-z) \begin{vmatrix} 0 & 1 & x^2+xy+y^2 \\ 0 & 1 & y^2+yz+z^2 \\ 1 & z & z^3 \end{vmatrix}$

$D = (x-y)(y-z)[y^2+yz+z^2-x^2-xy-y^2]$

$D = (x-y)(y-z)[y(z-x)+z^2-x^2]$

$D = (x-y)(y-z)(z-x)(x+y+z)$

(3) $\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Proof :

Let $D = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & y^4 \end{vmatrix}$

Apply $R_1 \to xR_1$; $R_2 \to yR_2, R_3 \to zR_3$ divide by xyz balancing

$D = \dfrac{1}{xyz} \begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix}$

$= \dfrac{xyz}{xyz} \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$.

$= \begin{vmatrix} 0 & x^2-y^2 & x^3-y^3 \\ 0 & y^2-z^2 & y^3-z^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

$= (x-y)(y-z)(xy+yz+zy)$

(4) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) < 0$ if a, b, c) are different and positive

proof :

$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = [bc-a^2]-[b^2-ac]+c(ab-c^2)$

$= 3abc - (a^3+b^3+c^3)$.

Solved Examples

Ex.4 The value of the determinant $\begin{vmatrix} 19 & 6 & 7 \\ 21 & 3 & 15 \\ 28 & 11 & 6 \end{vmatrix}$ is-

Sol. Applying $C_1 \to C_1 - (C_2 + C_3)$ we get Det.

$= \begin{vmatrix} 6 & 6 & 7 \\ 3 & 3 & 15 \\ 11 & 11 & 6 \end{vmatrix} = 0\ (\because\ C_1 = C_2)$

Ex.5 $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b+c-a & c+a-b & a+b-c \end{vmatrix}$ equals -

Sol. Determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ -(a+b+c) & -(a+b+c) & -(a+b+c) \end{vmatrix}$$

Applying $[R_3 + (-2R_2)]$,

We get $= -(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ 1 & 1 & 1 \end{vmatrix} = 0$

Ex.6 If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix}$ equals-

Sol. We know that if any row of a determinant is multiplied by k, then the value of the determinant is also multiplied by k. Here all the three rows are multiplied by k, therefore the value of new determinant will be $k^3\,\Delta$.

Ex.7 Simplify $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol. Let $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

$= (a+b+c)\,((b-c)(a-b) - (c-a)^2)$

$= (a+b+c)\,(ab + bc - ca - b^2 - c^2 + 2ca - a^2)$

$= (a+b+c)\,(ab + bc + ca - a^2 - b^2 - c^2)$

$\equiv 3abc - a^3 - b^3 - c^3$

Ex.8 Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Sol. Given detereminant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a-b)(b-c)\;[ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 - a^2b - a^2c - ab^2 - abc - b^3 - b^2c]$

$= (a-b)(b-c)\,[c(ab+bc+ca) - a(ab+bc+ca)]$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

Ex.9 The value of the determinant $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ is-

Sol. Applying $C_1 - C_2$ and $C_2 - C_3$, we get

$$\text{Det.} = \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix} = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$

[by $C_1 - C_2$, $C_3 - 10\,C_2$]

$$= \begin{vmatrix} 4 & 21 & 9 \\ 0 & 90 & -45 \\ 0 & -4 & 2 \end{vmatrix} \text{[By } R_2 + 3R_1,\ R_3 - R_1\text{]}$$

$= 4\,(180 - 180) = 0$

Ex.10 If x, y, z are unequal and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

then the value of xyz is-

Sol. Writing the given determinant as the sum of two determinants, we have

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} (1 + xyz) = 0$$

$\Rightarrow (x - y)(y - z)(z - x)(1 + xyz) = 0$

$\Rightarrow 1 + xyz = 0 \quad (\because x \neq y \neq z)$

$\Rightarrow xyz = -1$

Ex.11 The determinant $\begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-a)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix}$ is equal to -

Sol. Expanding the det., we get

$\Delta = -(b-a)^2 [0 - (a-c)^2 (c-b)^2] + (c-a)^2 [(a-b)^2 (b-c)^2 - 0]$

$= 2(a-b)^2 (b-c)^2 (c-a)^2.$

MULTIPLICATION OF TWO DETERMINANTS

1. Row by Row multiplication:

(ith row of Δ_1) × (jth row Δ_2) $\Delta_1\Delta_2$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

2. Row by Column Multiplication :

(ith row of Δ_1) × (jth Column Δ_2) $\Delta_1\Delta_2$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 & a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 & a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_3 & a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

3. Column by Row Multiplication :

(ith Column of Δ_1) × (jth Row Δ_2) $\Delta_1\Delta_2$

$$= \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1 & a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2 & a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3 \\ b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1 & b_1\alpha_2 + b_2\beta_2 + b_3\gamma_2 & b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3 \\ c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1 & c_1\alpha_2 + c_2\beta_2 + c_3\gamma_2 & c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3 \end{vmatrix}$$

4. Column by Column Multipliation :

(ith Column of Δ_1) × (jth Row Δ_2) $\Delta_1\Delta_2$

$$= \begin{vmatrix} a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 & a_1\beta_1 + a_2\beta_2 + a_3\beta_3 & a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 \\ b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 & b_1\beta_1 + b_2\beta_2 + b_3\beta_3 & b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3 \\ c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 & c_1\beta_1 + c_2\beta_2 + c_3\beta_3 & c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

But we prefer row by column multiplication.

To express a determinants as a product of two determinants :

To express a deteminant as product of two detenninants one requires a lot of practice and this can be done only by inspection and trial. It can be understood by the following examples.

Solved Examples

Ex.12 Let $\Delta = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$, then Δ can be expressed as

(a) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$ (b) $\begin{vmatrix} c & b & a \\ a & b & c \\ c & a & b \end{vmatrix}^2$

(c) $\begin{vmatrix} a & b & c \\ c & b & a \\ c & a & b \end{vmatrix}^2$ (d) None

Sol. $\Delta = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$

$$= \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix} = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$\left(\because \text{ by properties } \begin{vmatrix} c & a & b \\ b & c & a \\ -a & -b & -c \end{vmatrix} = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}\right)$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

Ex.13 If in the multiplication of $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}$ and $\begin{vmatrix} c & d \\ -d & c \end{vmatrix}$, A, B are the elements of the first row then the elements of the second row will be -

(A) –B, A (B) A, B

(C) B, A (D) –B, –A

Sol. $\begin{vmatrix} a & b \\ -b & a \end{vmatrix}\begin{vmatrix} c & d \\ -d & c \end{vmatrix} = \begin{vmatrix} ac+bd & -ad+bc \\ -bc+ad & bd+ac \end{vmatrix}$

$= \begin{vmatrix} ac+bd & bc+ad \\ -(bc-ad) & ac+bd \end{vmatrix} = \begin{vmatrix} A & B \\ -B & A \end{vmatrix}$

$\therefore$ required elements are –B, A. **Ans [A]**

Ex.14 If α, β are the roots of the equation, $ax^2 + bx + c = 0$ and $s_n = \alpha^n + \beta^n$.

Evaluate $\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1-s_2 & 1+s_3 & 1+s_4 \end{vmatrix}$ in terms of a, b, and c only.

Sol. $\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$

$= ((1-\alpha)(\alpha-\beta)(\beta-1))^2$

$= (1-(\alpha+\beta)+\alpha\beta)^2\ (\alpha-\beta)^2$

$= \left(1+\frac{b}{a}+\frac{c}{a}\right)^2 \left[\frac{b^2}{c^2}-\frac{4c}{a}\right]$

$= \frac{(a+b+c)^2\ (b^2-4ac)}{a^4}$

(ix) Limit of a determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then $\lim\limits_{x\to a} \Delta(x)$

$= \begin{vmatrix} \lim\limits_{x\to a} f(x) & \lim\limits_{x\to a} g(x) & \lim\limits_{x\to a} h(x) \\ \lim\limits_{x\to a} \ell(x) & \lim\limits_{x\to a} m(x) & \lim\limits_{x\to a} m(x) \\ \lim\limits_{x\to a} u(x) & \lim\limits_{x\to a} v(x) & \lim\limits_{x\to a} w(x) \end{vmatrix}$

provided each of nine limiting values exist finitely.

(x) Differentiation of a determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

then $\Delta'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} +$

$\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$

(xi) Integration of a Determinant

Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ \ell & m & n \end{vmatrix}$ where a, b, c, ℓ, m and n are constants

$\Rightarrow \int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ a & b & c \\ l & m & n \end{vmatrix}$

Solved Examples

Exa.15 If $\Delta(x) = \begin{vmatrix} \alpha+x & \theta+x & \lambda+x \\ \beta+x & \phi+x & \mu+x \\ \gamma+x & \psi+x & \nu+x \end{vmatrix}$.

show that $\Delta''(x) = 0$ and $\Delta(x) = \Delta(0) + Sx$, where S denote the sum of all the cofactors of all elements in $\Delta(0)$ and dash denotes the derivative with respect to x.

Sol. $\Delta'(x) = \begin{vmatrix} 1 & \theta+x & \lambda+x \\ 1 & \phi+x & \mu+x \\ 1 & \psi+x & \nu+x \end{vmatrix} + \begin{vmatrix} \alpha+x & 1 & \lambda+x \\ \beta+x & 1 & \mu+x \\ \gamma+x & 1 & \nu+x \end{vmatrix} +$

$\begin{vmatrix} \alpha+x & \theta+x & 1 \\ \beta+x & \phi+x & 1 \\ \nu+x & \psi+x & 1 \end{vmatrix}$

Applying $C_2 \to C_2 - x\,C_1$ and $C_3 \to C_3 - x\,C_3$ in first and $C_1 \to C_1 - x\,C_2$, $C_3 \to C_3 - x\,C_2$ in second and $C_1 \to C_1 - x\,C_3$ and $C_2 \to C_2 - x\,C_3$ in third to get,

$\Delta'(x) = \begin{vmatrix} 1 & \theta & \lambda \\ 1 & \phi & \mu \\ 1 & \psi & \nu \end{vmatrix} + \begin{vmatrix} \alpha & 1 & \lambda \\ \beta & 1 & \mu \\ \gamma & 1 & \nu \end{vmatrix} + \begin{vmatrix} \alpha & \theta & 1 \\ \beta & \phi & 1 \\ \gamma & \psi & 1 \end{vmatrix}$

$\Rightarrow \Delta'(x) = 0$

If S is the sum of all the cofactors of all elements in $\Delta(0)$, then it can be seen that $\Delta'(x) = S$.

on integrating $\Delta(x) = Sx + c$.

$\Rightarrow \Delta(0) = 0 + c$

Hence $\Delta(x) = Sx + \Delta(0)$.

Ex.16 Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$.

The value of $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$ is equal to :

(A) 1 (B) – 1

(C) 0 (D) None of these

Sol. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = f'(0)$.

$$f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$$

Thus $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

$+ \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$

Hence (C) is correct.

APPLICATION OF DETERMINANTS :

Following examples of short hand writing large expressions are:

(i) Area of a triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is:

$D = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If $D = 0$ then the three points are collinear.

(ii) Equation of a straight line passing through (x_1, y_1) & (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(iii) The lines: $a_1x + b_1y + c_1 = 0$........ (1)

$a_2x + b_2y + c_2 = 0$........ (2)

$a_3x + b_3y + c_3 = 0$........ (3)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Condition for the consistency of three simultaneous linear equations in 2 variables.

(iv) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

SOLVING A SYSTEM OF LINEAR EQUATIONS USING DETERMINANTS (CRAMER'S RULE)

CASE- I

NON-HOMOGENEOUS SYSTEM OF EQUATIONS

Consider three linear simultaneous equation in 'x', 'y', z'

$a_1x + b_1y + c_1z = d_1$ ...(i)

$a_2x + b_2y + c_2z = d_2$ ...(ii)

$a_3x + b_3y + c_3z = d_3$ ...(iii)

and

if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

then using Crammer's rule of determinant we get

$$\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}$$

i.e. $x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{\Delta_2}{\Delta}$, $z = \dfrac{\Delta_3}{\Delta}$

A. If $\Delta \neq 0$

Then $x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{\Delta_2}{\Delta}$, $z = \dfrac{\Delta_3}{\Delta}$

$\therefore$ The system is consistent and has unique solutions

B. If $\Delta = 0$ and

(i) If at least one of Δ_1, Δ_2, Δ_3 is not zero then the system of equations is inconsistent i.e. has no solution

(ii) If $d_1 = d_2 = d_3 = 0$ or $\Delta_1, \Delta_2, \Delta_3$ are all zero then the system of equations has infinitely many solutions.

Note :

In case of 3 planes are parallel

say $x + y + z = 2$

$2x + 2y + 2z = 1$

$4x + 4y + 4z = 3$

then the system is inconsistent & will be having no- solution, although $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

CASE- II

HOMOGENEOUS SYSTEM OF EQUATIONS

Consider three linear simultaneous equation in 'x', 'y', z'

$a_1x + b_1y + c_1z = 0$ ...(i)

$a_2x + b_2y + c_2z = 0$ ...(ii)

$a_3x + b_3y + c_3z = 0$ ...(iii)

and

$$\text{if } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \ \Delta_1 = \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0,$$

$$\Delta_2 = \begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0, \ \Delta_3 = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$$

$$\text{i.e. } x = \frac{\Delta_1}{\Delta} = \frac{0}{\Delta} = 0, \quad y = \frac{\Delta_2}{\Delta} = \frac{0}{\Delta} = 0,$$

$$z = \frac{\Delta_3}{\Delta} = \frac{0}{\Delta} = 0$$

A. If $\Delta \neq 0$ & $\Delta_1 = \Delta_2 = \Delta_3 = 0$

$x = y = z = 0$,

System is consistent & has unique solution which is called Trivial Solution or Zero Solution.

B. If $\Delta = 0$ and

$\Delta_1 = \Delta_2 = \Delta_3 = 0$ are all zero then the system of equations has infinitely many solutions which is called Non-Trivial Solution or Non-Zero Solution.

Solved Examples

Ex.17 Solve the system $\begin{matrix} x + y + z = 6 \\ x - y + z = 2 \\ 2x + y - z = 1 \end{matrix}$ using matrix inverse.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$.

Then the system is $AX = B$.

$|A| = 6$. Hence A is non singular.

$$\text{Cofactor } A = \begin{bmatrix} 0 & 3 & 3 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix} \quad \text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{6} \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix}$$

$$X = A^{-1} B = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/6 & -1/3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{i.e. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2, z = 3$.