TRIGONOMETRIC EQUATION

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An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation. e.g. $\cos^2 x - 4\sin x = 1$. It is noted that a trigonometrical identity is satisfied for every value of the unknown angle, whereas a trigonometric equation is satisfied only for some values (finite or infinite in number) of unknown angle, e.g., $\sin^2 x + \cos^2 x = 1$ is a trigonometrical identity as it is satisfied for every value of $x \in R$.

Solution or Root of a Trigonometric Equation

The value of an unknown angle which satisfies the given trigonometric equation is called a solution or root of the equation. For example, consider equation $2 \sin \theta = 1$. Clearly, $\theta = 30°$ and $\theta = 150°$ satisfy the equation ; therefore, 30° and 150° are solutions of the equation $2 \sin \theta = 1$ between 0° and 360°.

Principal Solution of a Trigonometric Equation

The solution of a trigonometric equation lie in the interval $[0, 2\pi)$. For example, if $\sin\theta = 1/2$, then the two values of θ between 0 and 2π are $\pi/6$ and $5\pi/6$. Thus, $\pi/6$ and $5\pi/6$. Thus, $\pi/6$ and $5\pi/6$ are the principal solution of equation $\sin \theta = 1/2$.

General Solution of a Trigonometric Equation

Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

Particular Solution of a Trigonometric Equation

The solution of the trigonometric equation lying in the given interval.

SOME IMPORTANT GENERAL SOLUTIONS OF EQUATIONS

Equation	Solution
$\sin\theta = 0$	$\theta = n\pi, n \in I$
$\cos\theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in I$
$\tan\theta = 0$	$\theta = n\pi, n \in I$
$\sin\theta = \sin\alpha$	$\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], n \in I$
$\cos\theta = \cos\alpha$	$\theta = 2n\pi \pm \alpha, n \in I, a \in [0,\pi]$
$\tan\theta = \tan\alpha$	$\theta = n\pi + \alpha, n \in I, \alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\sin\theta = 1$	$\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$
$\cos\theta = 1$	$\theta = 2n\pi, n \in I$
$\sin2\theta = \sin2\alpha$ or $\cos2\theta = \cos2\alpha$ or $\tan2\theta = \tan2\alpha$	$\theta = n\pi \pm \alpha, n \in I$

Important Points to be Remembered while Solving Trigonometric Equations

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

- **(i)** Check that denominator is not zero at any stage while solving equations.
- **(ii)** If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.
- **(iii)** If $\cot\theta$ or $\mathrm{cosec}\,\theta$ is involved in the equation, θ should not be multiple of π or 0.

Ex. Solve the equation $\sin x + \cos x = 1$.

Sol. Squaring the given equation, we get

$(\sin x + \cos x)^2 = 1$

or $1 + \sin 2x = 1$

or $\sin 2x = 0$

$\Rightarrow$ $2x = n\pi, n \in Z$

or $x = n\pi/2, n \in Z$

But for $n = 2, 6, 10, ...$

$\sin x + \cos + = -1$, which contradicts the given equation

Also,for $x = 3, 7, 11, ...$

$\sin x + \cos x = -1$

Hence, the solution is $x = 2n\pi$ or $x = (4n + 1)\frac{\pi}{2}$.

Ex. Solve $4\tan^2\theta = 3\sec^2\theta$

Sol. ✈ $4\tan^2\theta = 3\sec^2\theta$ ......**(i)**

For equation **(i)** to be defined $\theta \ne (2n+1)\frac{\pi}{2}, n \in I$

✈ equation **(i)** can be written as:

$\frac{4\sin^2\theta}{\cos^2\theta} = \frac{3}{\cos^2\theta}$ ✈ $\theta \ne (2n+1)\frac{\pi}{2}, n \in I$

$\therefore$ $\cos^2\theta \ne 0$

$\Rightarrow$ $4\sin^2\theta = 3$ $\Rightarrow$ $\sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow$ $\sin^2\theta = \sin^2\frac{\pi}{3}$ $\Rightarrow$ $\theta = n\pi \pm \frac{\pi}{3}, n \in I$

Ex. Find the set of values of x for which $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x.\tan 2x} = 1$.

Sol. We have, $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x.\tan 2x} = 1$ $\Rightarrow$ $\tan(3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow$ $\tan x = \tan\dfrac{\pi}{4} \Rightarrow x = n\pi + \dfrac{\pi}{4}, n \in I$ {using $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$}

But for this value of x, tan 2x is not defined.

Hence the solution set for x is ϕ.

Ex. Find the most general solution of $2^{1+|\cos x|+\cos^2 x+|\cos x|^3+....\infty} = 4$

Sol. We have, $2^{1+|\cos x|+\cos^2 x+|\cos x|^3+....\infty} = 4$

or $2^{1+|\cos x|+\cos^2 x+|\cos x|^3+....\infty} = 4$

or $2^{\frac{1}{1-|\cos x|}} = 2^2$

or $\dfrac{1}{1-|\cos x|} = 2$

or $|\cos x| = \dfrac{1}{2}$ or $\cos x = \pm\dfrac{1}{2}$

$x = n\pi \pm \dfrac{\pi}{3}, n \in Z$

TYPES OF TRIGONOMETRIC EQUATIONS

(I) Solving Trigonometric Equations by Factorization

Ex. Solve $(2 \sin x - \cos x)(1 + \cos x) = \sin^2 x$

Sol. $(2 \sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore$ $(1 + \cos x)(2 \sin x - \cos x - 1 + \cos x) = 0$

$\therefore$ $(1 + \cos x)(2 \sin x - 1) = 0$

$\Rightarrow$ $\cos x = -1$ or $\sin x = \dfrac{1}{2}$

$\Rightarrow$ $\cos x = -1 = \cos\pi$ $\Rightarrow$ $x = 2n\pi + \pi = (2n + 1)\pi, n \in I$

or $\sin x = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$ $\Rightarrow$ $x = k\pi + (-1)^k \dfrac{\pi}{6}, k \in I$

Ex. If $\dfrac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then find the general solution for θ.

Sol. Since, $\dfrac{1}{6}\sin\theta, \cos\theta, \tan\theta$ are in G.P.

$\Rightarrow$ $\cos^2\theta = \dfrac{1}{6}\sin\theta . \tan\theta$ $\Rightarrow$ $6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore$ $(2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow$ $\cos\theta = \dfrac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow$ $\cos\theta = \cos\dfrac{\pi}{3}$ $\Rightarrow$ $\theta = 2n\pi \pm \dfrac{\pi}{3}, n \in I.$

(II) Solutions of Equations Reducible to Quadratic Equations.

Ex. $6 - 10\cos x = 3\sin^2 x$

Sol. $6 - 10\cos x = 3 - 3\cos^2 x \quad \Rightarrow \quad 3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow \quad (3\cos x - 1)(\cos x - 3) = 0 \quad \Rightarrow \quad \cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore \quad \cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right) \Rightarrow \quad x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right), n \in I$

Ex. Find the number of solutions of $\sin^2 x - \sin x - 1 = 0$ in $[-2\pi, 2\pi]$

Sol. $\sin^2 x - \sin x - 1 = 0$

$\sin x = \frac{1 \pm \sqrt{5}}{2}$

$= \frac{1 - \sqrt{5}}{2} \qquad$ [$\sin x$ $\frac{1+\sqrt{5}}{2} > 1$ not possible]

Hence, x can attain two values in $[0, 2\pi]$ and two more values in $[-2\pi, 0)$. Thus, there are four solutions.

Ex. Find the number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$.

Sol. Here, $\tan x + \sec x = 2\cos x \quad \Rightarrow \quad \sin x + 1 = 2\cos^2 x$

$\Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0 \quad \Rightarrow \quad \sin x = \frac{1}{2}, -1$

But $\sin x = -1 \quad \Rightarrow \quad x = \frac{3\pi}{2}$ for which $\tan x + \sec x = 2\cos x$ is not defined.

Thus $\sin x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ number of solutions of $\tan x + \sec x = 2\cos x$ is 2.

(III) Trigonometric Equations which can be Solved by Transforming a Product of Trigonometric Ratios into their Sum or Difference

Ex. Solve $\sin 5x.\cos 3x = \sin 6x.\cos 2x$

Sol $\Rightarrow \quad \sin 5x.\cos 3x = \sin 6x.\cos 2x \quad \Rightarrow \quad 2\sin 5x.\cos 3x = 2\sin 6x.\cos 2x$

$\Rightarrow \quad \sin 8x + \sin 2x = \sin 8x + \sin 4x \quad \Rightarrow \quad \sin 4x - \sin 2x = 0$

$\Rightarrow \quad 2\sin 2x.\cos 2x - \sin 2x = 0 \quad \Rightarrow \quad \sin 2x\,(2\cos 2x - 1) = 0$

$\Rightarrow \quad \sin 2x = 0 \quad$ or $\quad 2\cos 2x - 1 = 0$

$\Rightarrow \quad 2x = n\pi, n \in I \quad$ or $\quad \cos 2x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{n\pi}{2}, n \in I \quad$ or $\quad 2x = 2n\pi \pm \frac{\pi}{3}, n \in I$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is

$\frac{n\pi}{2}, n \in I \quad$ or $\quad n\pi \pm \frac{\pi}{6}, n \in I$

(IV) Solution of Equations of Form acosθ + bsinθ = c

To solve equation, let us convert the equation to the form cos θ = cos α or sinθ = sin α.

For this let us consider

$$\left.\begin{array}{l} a = r\cos\phi \\ b = r\sin\phi \end{array}\right\} \Rightarrow \left\{\begin{array}{l} r = \sqrt{a^2+b^2} \\ \tan\phi = \dfrac{b}{a} \end{array}\right.$$

Substituting these values in the equation a cos θ + b sin θ = c, we have

$$r\cos\phi\cos\theta + r\sin\phi\sin\theta = c$$

or $\quad r\cos(\theta-\phi) = c$

or $\quad \cos(\theta-\phi) = \dfrac{c}{r} = \dfrac{c}{\sqrt{a^2+b^2}} = \cos\beta$

$\Rightarrow \quad \theta - \phi = 2n\pi \pm \beta$

$\Rightarrow \quad \theta = 2n\pi + \phi \pm \beta,\ n \in Z$

Here ϕ and β are calculated if a, b and c are given.

Hence, we can solve the equation of this type by putting

$a = r\cos\phi$ and $\beta = r\sin\phi$, provided $\left|\dfrac{c}{\sqrt{a^2+b^2}}\right| \le 1$ [∵ cos β lies between – 1 and 1]

or $\quad \left|\dfrac{c}{\sqrt{a^2+b^2}}\right| \le 1 \quad$ or $\quad |c| \le \sqrt{a^2+b^2}$

Working Rule

1. First of all check whether $|c| \le \sqrt{a^2+b^2}$ or not.
2. If $|c| > \sqrt{a^2+b^2}$, then the given equation has no real solution.
3. If $|c| \le \sqrt{a^2+b^2}$, then divide both sides of the equation by $\sqrt{a^2+b^2}$.
4. Take $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$, then the given equation will become $\cos(\theta-\alpha) = \cos\beta$,

   where $\tan\alpha = \dfrac{b}{a}$ and $\cos\beta = \dfrac{c}{\sqrt{a^2+b^2}}$.

   We can also take $\sin\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$, $\cos\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$, and $\sin\beta = \dfrac{c}{\sqrt{a^2+b^2}}$, then the given equation will reduce to the form $\sin(\theta+\alpha) = \sin\beta$.

Ex. Solve $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta\cos 3\theta$.

Sol. We have, $\sqrt{3}\cos\theta - 3\sin\theta = 2(\sin 5\theta - \sin\theta)$.

or $\left(\frac{\sqrt{3}}{2}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta = \sin 5\theta$

$\Rightarrow$ $\cos\left(\theta + \frac{\pi}{6}\right) = \sin 5\theta = \cos\left(\frac{\pi}{2} - 5\theta\right)$

$\Rightarrow$ $\theta + \frac{\pi}{6} = 2n\pi \pm \left(\frac{\pi}{2} - 5\theta\right)$

$\theta = \frac{n\pi}{3} + \frac{\pi}{18}$ or $\theta = -\frac{n\pi}{2} + \frac{\pi}{6}$, $\forall\ n \in Z$

Ex. Prove that the equation $k\cos x - 3\sin x = k + 1$ possess a solution iff $k \in (-\infty, 4]$.

Sol. Here, $k\cos x - 3\sin x = k + 1$, could be re-written as :

$$\frac{k}{\sqrt{k^2+9}}\cos x - \frac{3}{\sqrt{k^2+9}}\sin x = \frac{k+1}{\sqrt{k^2+9}}$$

or $\cos(x+\phi) = \frac{k+1}{\sqrt{k^2+9}}$, where $\tan\phi = \frac{3}{k}$

which possess a solution only if $-1 \le \frac{k+1}{\sqrt{k^2+9}} \le 1$

i.e., $\left|\frac{k+1}{\sqrt{k^2+9}}\right| \le 1$

i.e., $(k+1)^2 \le k^2 + 9$

i.e., $k^2 + 2k + 1 \le k^2 + 9$

or $k \le 4$

$\Rightarrow$ The interval of k for which the equation ($k\cos x - 3\sin x = k + 1$) has a solution is $(-\infty, 4]$.

(V) Solving Trigonometric Equations by Transforming sum of Trigonometric Functions into Product

Ex. Solve $\cos 3x + \sin 2x - \sin 4x = 0$

Sol. $\cos 3x + \sin 2x - \sin 4x = 0$

$\Rightarrow$ $\cos 3x + 2\cos 3x.\sin(-x) = 0$

$\Rightarrow$ $\cos 3x - 2\cos 3x.\sin x = 0$

$\Rightarrow$ $\cos 3x(1 - 2\sin x) = 0$

$\Rightarrow$ $\cos 3x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow$ $3x = (2n+1)\frac{\pi}{2}, n \in I$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{6}, n \in I$ or $x = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

$\therefore$ solution of given equation is

$(2n+1)\frac{\pi}{6}, n \in I$ or $n\pi + (-1)^n\frac{\pi}{6}, n \in I$

(VI) Solving Equations by a Change of Variable

(i) Equations of the form P (sin x ± cos x, sin x. cos x) = 0, where P (y,z) is a polynomial, can be solved by the substitution $\cos x \pm \sin x = t \Rightarrow 1 \pm 2\sin x.\cos x = t^2$.

e.g. $\sin x + \cos x = 1 + \sin x.\cos x$.

put $\sin x + \cos x = t$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow 2\sin x\cos x = t^2 - 1 \quad (\because \sin^2 x + \cos^2 x = 1)$

$\Rightarrow \sin x.\cos x = \left(\frac{t^2 - 1}{2}\right)$

Substituting above result in given equation, we get :

$$t = 1 + \frac{t^2 - 1}{2}$$

$\Rightarrow 2t = t^2 + 1 \quad \Rightarrow \quad t^2 - 2t + 1 = 0$

$\Rightarrow (t - 1)^2 = 0 \quad \Rightarrow \quad t = 1$

$\Rightarrow \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2 + 1^2}$ i.e. $\sqrt{2}$, we get

$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \cos x\cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad \Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2} = (4n + 1)\frac{\pi}{2}$, $n \in I$

(ii) Equations of the form of $a\sin x + b\cos x + d = 0$, where a, b & d are real numbers can be solved by changing sin x & cos x into their corresponding tangent of half the angle.

e.g. $3\cos x + 4\sin x = 5$

$\Rightarrow 3\left(\frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}\right) + 4\left(\frac{2\tan x/2}{1 + \tan^2 x/2}\right) = 5$

$\Rightarrow \frac{3 - 3\tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + \frac{8\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = 5$

$\Rightarrow 3 - 3\tan^2\frac{x}{2} + 8\tan\frac{x}{2} = 5 + 5\tan^2\frac{x}{2} \quad \Rightarrow \quad 8\tan^2\frac{x}{2} - 8\tan\frac{x}{2} + 2 = 0$

$\Rightarrow 4\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + 1 = 0 \quad \Rightarrow \quad \left(2\tan\frac{x}{2} - 1\right)^2 = 0$

$\Rightarrow 2\tan\frac{x}{2} - 1 = 0 \quad \Rightarrow \quad \tan\frac{x}{2} = \frac{1}{2} = \tan\left(\tan^{-1}\frac{1}{2}\right)$

$\Rightarrow \frac{x}{2} = n\pi + \tan^{-1}\left(\frac{1}{2}\right), n \in I \quad \Rightarrow \quad x = 2n\pi + 2\tan^{-1}\frac{1}{2}, n \in I$

(iii) Many equations can be solved by introducing a new variable.

e.g. $\sin^4 2x + \cos^4 2x = \sin 2x . \cos 2x$

substituting $\sin 2x . \cos 2x = y$

$\Rightarrow$ $(\sin^2 2x + \cos^2 2x)^2 = \sin^4 2x + \cos^4 2x + 2\sin^2 2x.\cos^2 2x$

$\Rightarrow$ $\sin^4 2x + \cos^4 2x = 1 - 2\sin^2 2x.\cos^2 2x$ substituting above result in given equation :

$1 - 2y^2 = y$

$\Rightarrow$ $2y^2 + y - 1 = 0$ $\Rightarrow$ $2(y+1)\left(y - \frac{1}{2}\right) = 0$

$\Rightarrow$ $y = -1$ or $y = \frac{1}{2}$ $\Rightarrow$ $\sin 2x.\cos 2x = -1$ or $\sin 2x.\cos 2x = \frac{1}{2}$

$\Rightarrow$ $2\sin 2x.\cos 2x = -2$ or $2\sin 2x.\cos 2x = 1$

$\Rightarrow$ $\sin 4x = -2$ (which is not possible) or $2\sin 2x.\cos 2x = 1$

$\Rightarrow$ $\sin 4x = 1 = \sin \frac{\pi}{2}$ $\Rightarrow$ $4x = n\pi + (-1)^n \frac{\pi}{2}, n \in I$ $\Rightarrow$ $x = \frac{n\pi}{4} + (-1)^n \frac{\pi}{8}, n \in I$

Ex. Find the general solution of equation $\sin^4 x + \cos^4 x = \sin x \cos x$.

Sol. Using half-angle formulae, we can represent given equation in the form :

$$\left(\frac{1-\cos 2x}{2}\right)^2 + \left(\frac{1+\cos 2x}{2}\right)^2 = \sin x \cos x$$

$\Rightarrow$ $(1 - \cos 2x)^2 + (1 + \cos 2x)^2 = 4\sin x \cos x$

$\Rightarrow$ $2(1 + \cos^2 2x) = 2\sin 2x$ $\Rightarrow$ $1 + 1 - \sin^2 2x = \sin 2x$

$\Rightarrow$ $\sin^2 2x + \sin 2x = 2$

$\Rightarrow$ $\sin 2x = 1$ or $\sin 2x = -2$ (which is not possible)

$\Rightarrow$ $2x = 2n\pi + \frac{\pi}{2}, n \in I$

$\Rightarrow$ $x = n\pi + \frac{\pi}{4}, n \in I$

(VII) Solving Trigonometric Equations with the Use of the Boundness of the Functions Involved.

Ex. Solve $\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right).\cos x = 0$

Sol. $\sin x \cos\frac{x}{4} + \cos x \sin\frac{x}{4} + \cos x = 2$

$\therefore$ $\sin\left(\frac{5x}{4}\right) + \cos x = 2$

$\Rightarrow$ $\sin\left(\frac{5x}{4}\right) = 1$ & $\cos x = 1$ (as $\sin\theta \le 1$ & $\cos\theta \le 1$)

Now consider

$\cos x = 1 \Rightarrow x = 2\pi, 4\pi, 6\pi, 8\pi$ ....

and $\sin\frac{5x}{4} = 1 \Rightarrow x = \frac{2\pi}{5}, \frac{10\pi}{5}, \frac{18\pi}{5}$ ....

Common solution to above APs will be the AP having

First term = 2π

Common difference = LCM of 2π and $\frac{8\pi}{5} = \frac{40\pi}{5} = 8\pi$

$\therefore$ General solution will be general term of this AP i.e. $2\pi + (8\pi)n, n \in I$

$\Rightarrow$ $x = 2(4n+1)\pi, n \in I$

Ex. Solve the equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $0 \le x \le \pi$.

Sol. We know, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ and $-1 \le \sin\theta \le 1$.

$\therefore$ $(\sin x + \cos x)$ admits the maximum value as $\sqrt{2}$

and $(1 + \sin 2x)$ admits the maximum value as 2.

Also $(\sqrt{2})^2 = 2$.

$\therefore$ the equation could hold only when, $\sin x + \cos x = \sqrt{2}$ and $1 + \sin 2x = 2$

Now, $\sin x + \cos x = \sqrt{2} \Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow$ $x = 2n\pi + \pi/4, n \in I$ ...... **(i)**

and $1 + \sin 2x = 2 \Rightarrow \sin 2x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow$ $2x = m\pi + (-1)^m\frac{\pi}{2}, m \in I \Rightarrow x = \frac{m\pi}{2} + (-1)^m\frac{\pi}{4}$ ...... **(ii)**

The value of x in $[0, \pi]$ satisfying equations **(i)** and **(ii)** is $x = \frac{\pi}{4}$ (when n = 0 & m = 0)

$\sin x + \cos x = -\sqrt{2}$ and $1 + \sin 2x = 2$ also satisfies but as $x \ge 0$, this solution is not in domain.

Ex. Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$

Sol. $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \le 1$ ...... **(i)**

$$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \le 1$$

Minimum value of $2^{\frac{1}{\cos^2 x}} = 2$

Minimum value of $\sqrt{\left(y-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + \frac{1}{2}}$ is 1

$\Rightarrow$ **(i)** is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

$\Rightarrow$ $\cos^2 x = 1$ and $y = 1/2$ $\Rightarrow$ $\cos x = \pm 1$ $\Rightarrow$ $x = n\pi$, where $n \in I$.

Hence $x = n\pi$, $n \in I$ and $y = 1/2$.

Ex. Find the number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$, $0 \le x \le \pi/2$.

Sol. Let $y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$ $\Rightarrow$ $y = (1 + \cos x)\sin^2 x$ and $y = x^2 + \frac{1}{x^2}$

when $y = (1 + \cos x)\sin^2 x = (\text{a number} < 2)(\text{a number} \le 1)$ $\Rightarrow$ $y < 2$ ...... **(i)**

and when $y = x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 \ge 2$ $\Rightarrow$ $y \ge 2$ ...... **(ii)**

No value of y can be obtained satisfying **(i)** and **(ii)**, simultaneously

$\Rightarrow$ No real solution of the equation exists.

If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different values of θ, then solution does not exist.

INEQUALITIES

Trigonometric Inequations

To solve the trigonometric inequation of the type $f(x) \le a$, or $f(x) \ge a$ where $f(x)$ is some trigonometric ratio, the following steps should be taken :

1. Draw the graph of $f(x)$ in an interval length equal to the fundamental period of $f(x)$.

2. Draw the line $y = a$.

3. Take the portion of the graph for which the inequation is satisfied.

4. To generalize, add nT ($n \in I$) and take union over the set of integers, where T is the fundamental period of $f(x)$.

Ex. Solve $2\cos^2\theta + \sin\theta \le 2$, where $\pi/2 \le \theta \le 3\pi/2$.

Sol. $2\cos^2\theta + \sin\theta \le 2$

or $2(1 - \sin^2\theta) + \sin\theta \le 2$

or $-2\sin^2\theta + \sin\theta \le 0$

or $2\sin^2\theta - \sin\theta \ge 0$

or $\sin\theta(2\sin\theta - 1) \ge 0$

or $\sin\theta(\sin\theta - 1/2) \ge 0$,

which is possible if $\sin\theta \le 0$ or $\sin\theta \ge 1/2$.

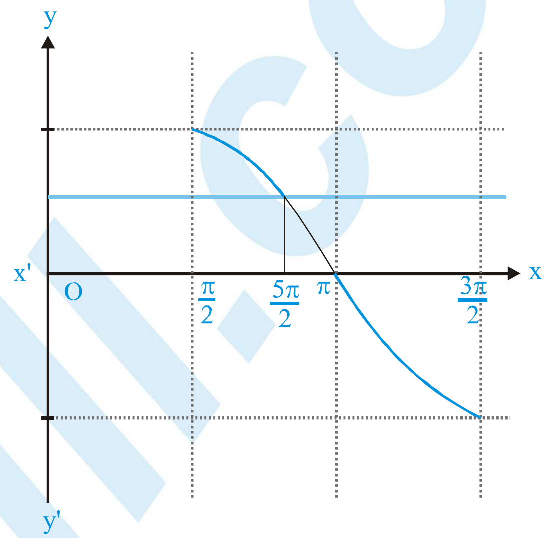
From the graph,

$\sin\theta \ge 1/2 \quad \Rightarrow \quad \pi/2 \le \theta \le 5\pi/6$

$\sin\theta \le 0 \quad \Rightarrow \quad \pi \le \theta \le 3\pi/2$

Hence, the required values of θ are given by

$\theta \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$



Ex. Sovle $\sin\theta + \sqrt{3}\cos\theta \ge 1, -\pi < \theta \le \pi$.

Sol. The given inequation is

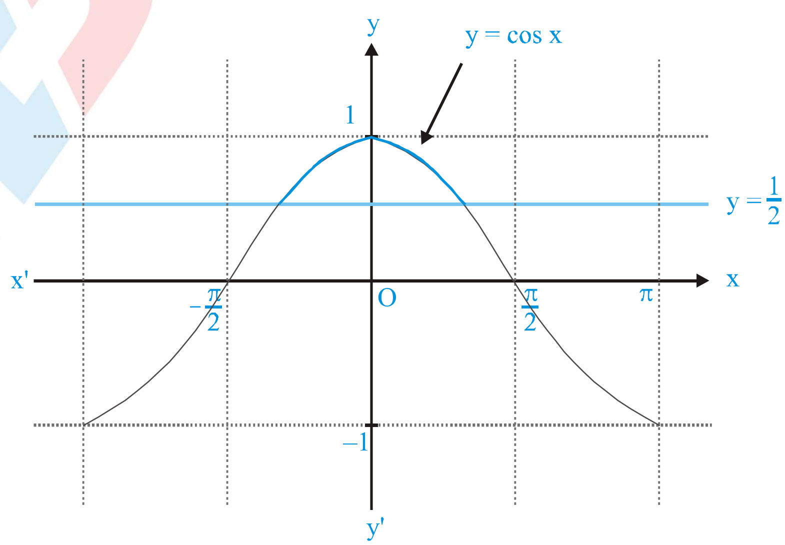
$\sin\theta + \sqrt{3}\cos\theta \ge 1, -\pi < \theta \le \pi$

$$\Rightarrow \quad \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta \ge \frac{1}{2}$$

$$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{6}\right) \ge \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad -\frac{\pi}{3} \le \theta - \frac{\pi}{6} \le \frac{\pi}{3}$$

$$\Rightarrow \quad -\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$$



1. Trigonometric Equation

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. Solution of Trigonometric Equation

A value of the unknown angle which satisfies the given equation is called a trigonometric equation.

(a) Principle Solution :- The solution of the trigonometric equation lying in the interval $[0, 2\pi]$.

(b) General Solution :- Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solutions of trigonometric equation.

3. General Solutions of Some Trigonometric Equations

(a) If $\sin\theta = 0$, then $\theta = n\pi, n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\dfrac{\pi}{2}\ n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi, n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha, n \in I, \alpha \in [0, \pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha, n \in I, \alpha \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \dfrac{\pi}{2} = (4n+1)\dfrac{\pi}{2}, n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi, n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha, n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n, n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$

$\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n, n \in I$

(l) If n is an odd integer then $\sin\dfrac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$, $\cos\ (-1)^{\frac{n-1}{2}}\ \cos\dfrac{n\pi}{2} = 0$

(m) $\sin\left(\dfrac{n\pi}{2} + 0\right) = (-1)^{\frac{n-1}{2}}\cos\theta$, $\cos\left(\dfrac{n\pi}{2} + 0\right) = (-1)^{\frac{n-1}{2}}\sin\theta$

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4. General Solution of Equation acosθ + bsinθ

Consider, $a \sin\theta + b\cos\theta = c$ ....(i)

$$\therefore \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has the solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi, \frac{b}{\sqrt{a^2+b^2}}\sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxiliary argument ϕ, equation (i) reduces to $\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$

Now this equation can be solved easily.

5. General Solution of Equation of Form

$a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + a_2 \sin^{n-2} \times \cos^2 x + \text{.........} + a_n \cos^n x = 0$

$a_0, a_1, \text{............} a_n$ are real numbers

Such an equation is solved by dividing equation by $\cos^n x$.

6. Important Tips

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may leads to extraneous solutions.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a products because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(e) Check that denominators is not zero at any stage while solving equations.

(f) **(i)** If tan θ or sec θ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$

(ii) If cot θ or cosec θ is involved in the equation, θ should not be integral multiple of π or 0.

(g) If two different trigonometric ratios such as tan θ and sec θ are involved then after solving we cannot apply the usual formulae for general solution because periodicity of the functions are not same.

(h) If L.H.S of the given trigonometric equation is always greater than k, then no solution exists. If both the sides are equal to k for s ame value of θ, then solution exists and if they are equal for different value of θ, then solution does not exist.