Inverse Trigonometric Functions

INTRODUCTION :

The student may be familiar about trigonometric functions viz sin x, cos x, tan x, cosec x, sec x, cot x with respective domains R, R, R – {(2n + 1) π/2}, R – {nπ}, R – {(2n + 1) π/2}, R – {nπ} and respective ranges [–1, 1], [–1, 1], R, R – (–1, 1), R – (–1, 1), R. Correspondingly, six inverse trigonometric functions (also called inverse circular functions) are defined.

$\sin^{-1}x$, $\cos^{-1}1x$, $\tan^{-1}1x$ etc. denote angles or real numbers whose sine is x , whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available. These are also written as arc sinx, arc cosx etc.

Let sinθ = x then θ = Arc sin x

$\because \quad -1 \le \sin\theta \le 1$ and $\sin\theta = x$

$\therefore \quad -1 \le x \le 1$

Thus, Arc sinx is defined only when $-1 \le x \le 1$

Clearly, for every $x \in [-1, 1]$, infinite number of values of Arc sinx will be obtained.

Thus, $\sin^{-1}x$ or arc sinx is the principal value of the angle whose sine is equal to x.

i.e.

(i) $\sin\theta = x \Leftrightarrow \sin^{-1} x = \theta$

(ii) $\cos\theta = x \Leftrightarrow \cos^{-1} x = \theta$

(iii) $\tan\theta = x \Leftrightarrow \tan^{-1} x = \theta$

(iv) $\cot\theta = x \Leftrightarrow \cot^{-1} x = \theta$

(v) $\sec\theta = x \Leftrightarrow \sec^{-1} x = \theta$

(vi) $\text{cosec}\theta = x \Leftrightarrow \text{cosec}^{-1}x = \theta$

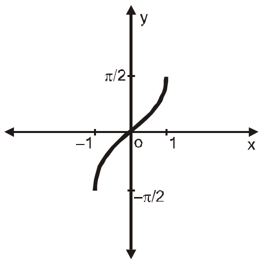
If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken.

$\sin^{-1}x$:

The symbol $\sin^{-1}x$ or arcsinx denotes the angle θ so that sin θ = x. As a direct meaning, $\sin^{-1}x$ is not a function, as it does not satisfy the requirements for a rule to become a function. But by a suitable choice [–1, 1] as its domain and standardized set [–π/2, π/2] as its range, then rule $\sin^{-1} x$ is a single valued function.

Thus $\sin^{-1}x$ is considered as a function with domain [–1, 1] and range [–π/2, π/2].

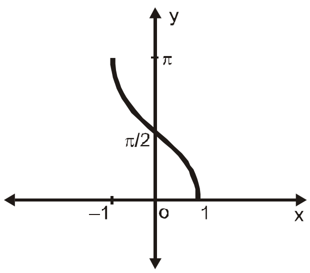
The graph of $y = \sin^{-1}x$ is as shown below, which is obtained by taking the mirror image, of the portion of the graph of y = sin x, from x = –π/2 to x = π/2, on the line y = x.



$\cos^{-1}x$:

By following the discussions, similar to above, we have $\cos^{-1}x$ or arccos x as a function with domain [–1, 1] and range [0, π].

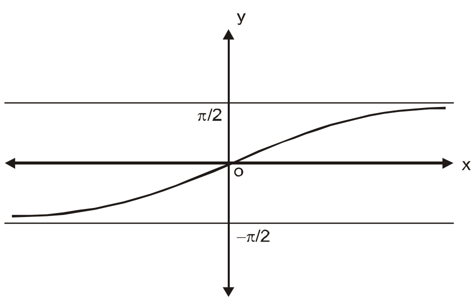
The graph of $y = \cos^{-1}x$ is similarly obtained as the mirror image of the portion of the graph of y = cos x from x = 0 to x = π.



$\tan^{-1}x$:

We get $\tan^{-1}x$ or arctanx as a function with domain R and range (–π/2, π/2).

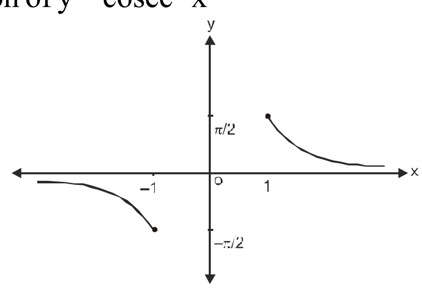
Graph of $y = \tan^{-1}x$



$\text{cosec}^{-1}x$:

$\text{cosec}^{-1}x$ or arccosec x is a function with domain R – (–1, 1) and range [–π/2, π/2] – {0}.

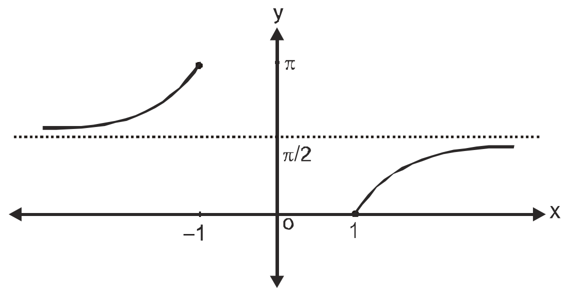
Graph of $y = \text{cosec}^{-1}x$



$\sec^{-1}x$:

$\sec^{-1}x$ or arcsec x is a function with domain R – (–1, 1) and range [0, π] – {π/2}.

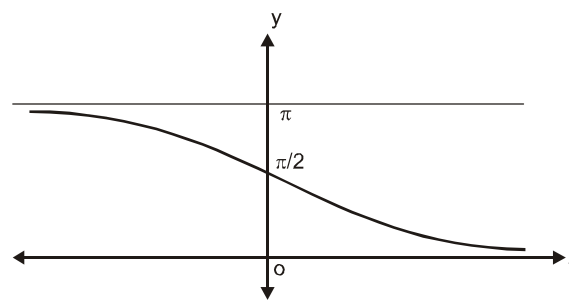
Graph of $y = \sec^{-1}x$



$\cot^{-1}x$:

$\cot^{-1}x$ or arccot x is a function with domain R and range (0, π)

Graph of $y = \cot^{-1}x$



DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

Function	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1,\infty)$	$\cup \left(\frac{\pi}{2}, \pi\right]$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1,\infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Note that :

(a) 1st quadrant is common to all the inverse functions.

(b) 3rd quadrant is **not used** in inverse functions.

(c) 4th quadrant is used in the **CLOCKWISE DIRECTION** i.e. $-\frac{\pi}{2} \le y \le 0$.

(d) $\sin^{-1} x]_{max} = \frac{\pi}{2}$ and $\sin^{-1} x]_{min} = -\frac{\pi}{2}$

$\cos^{-1} x]_{max} = \pi$ and $\cos^{-1} x]_{min} = 0$

Solved Examples

Ex.1 Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

Sol. $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \tan\left[\frac{\pi}{3} + \left(-\frac{\pi}{6}\right)\right]$

$= \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.

Ex.2 Find domain of $\sin^{-1}(2x^2 - 1)$

Sol. Let $y = \sin^{-1}(2x^2 - 1)$

For y to be defined $-1 \le (2x^2 - 1) \le 1$

$\Rightarrow 0 \le 2x^2 \le 2 \Rightarrow 0 \le x^2 \le 1 \Rightarrow x \in [-1, 1]$.

Property 1 : "–x"

The graphs of $\sin^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$ are symmetric about origin.

Hence we get $\sin^{-1}(-x) = -\sin^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$.

Also the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \pi/2)$. From this, we get $\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$.

Property 2 : $T(T^{-1})$

(i) $\sin(\sin^{-1}x) = x, \; -1 \le x \le 1$

Proof :

Let $\theta = \sin^{-1}x$. Then $x \in [-1, 1]$ & $\theta \in [-\pi/2, \pi/2]$.

$\Rightarrow$ $\sin\theta = x$, by meaning of the symbol

$\Rightarrow$ $\sin(\sin^{-1}x) = x$

Similar proofs can be carried out to obtain

(ii) $\cos(\cos^{-1}x) = x, \quad -1 \le x \le 1$

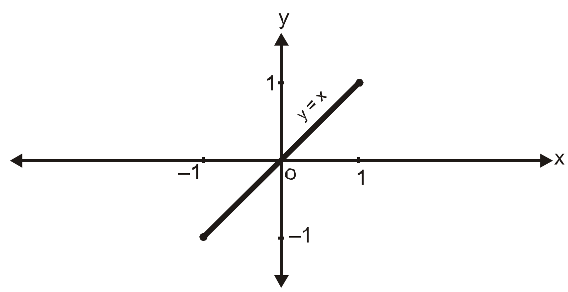
(iii) $\tan(\tan^{-1}x) = x, \quad x \in R$

(iv) $\cot(\cot^{-1}x) = x, \quad x \in R$

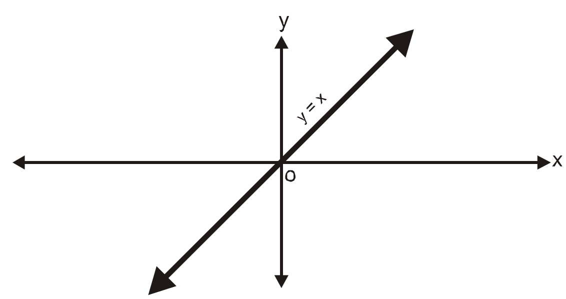
(v) $\sec(\sec^{-1}x) = x, \quad x \le -1, x \ge 1$

(vi) $\text{cosec}(\text{cosec}^{-1}x) = x, \quad |x| \ge 1$

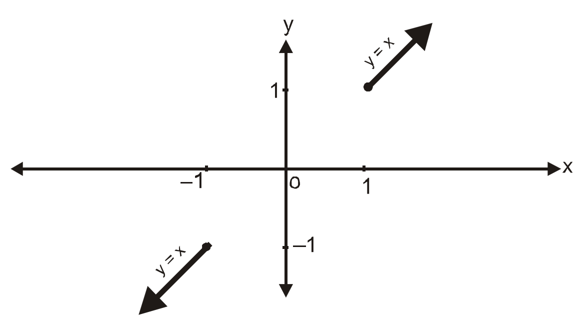
The graph of $y = \sin(\sin^{-1}x) \equiv \cos(\cos^{-1}x)$



The graph of $y = \tan(\tan^{-1}x) \equiv \cot(\cot^{-1}x)$



The graph of $y = \text{cosec}(\text{cosec}^{-1}x) \equiv \sec(\sec^{-1}x)$



Property 3 : $T^{-1}(T)$

(i) $\sin^{-1}(\sin x)$

$$= \begin{cases} -2n\pi + x, & x \in [2n\pi - \pi/2, 2n\pi + \pi/2] \\ (2n+1)\pi - x, & x \in [(2n+1)\pi - \pi/2, (2n+1)\pi + \pi/2], n \in Z \end{cases}$$

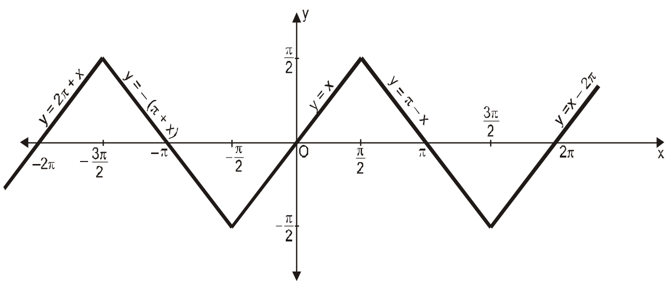
Proof :

If $x \in [2n\pi - \pi/2, 2n\pi + \pi/2]$, then $-2n\pi + x \in [-\pi/2, \pi/2]$ and $\sin(-2n\pi + x) = \sin x$.

Hence $\sin^{-1}(\sin x) = -2n\pi + x$ for $x \in [2n\pi - \pi/2, 2n\pi + \pi/2]$.

Proof of 2nd part is left for the students.

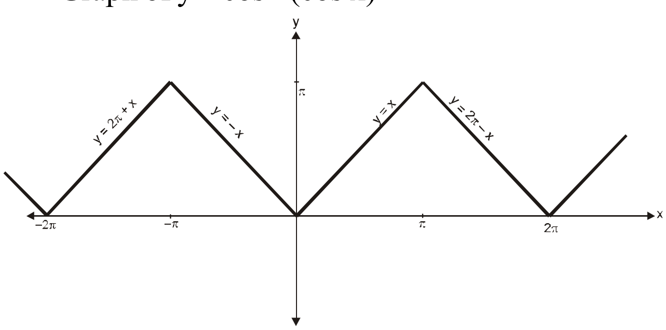
Graph of $y = \sin^{-1}(\sin x)$



(ii) $\cos^{-1}(\cos x)$

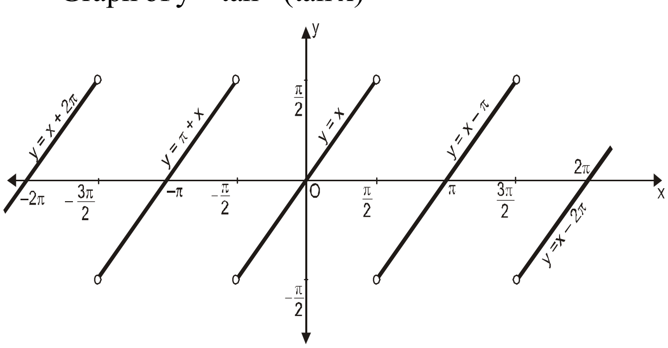
$$= \begin{cases} -2n\pi + x, & x \in [2n\pi, (2n+1)\pi] \\ 2n\pi - x, & x \in [(2n-1)\pi, 2n\pi], n \in I \end{cases}$$

Graph of $y = \cos^{-1}(\cos x)$



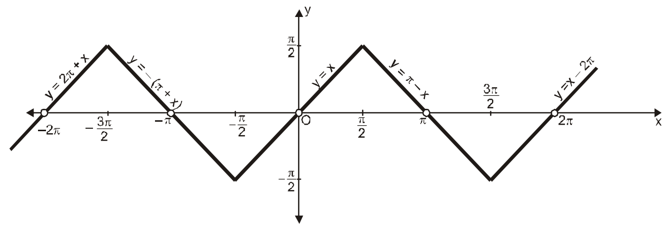
(iii) $\tan^{-1}(\tan x) = -n\pi + x$, $n\pi - \pi/2 < x < n\pi + \pi/2$, $n \in Z$

Graph of $y = \tan^{-1}(\tan x)$



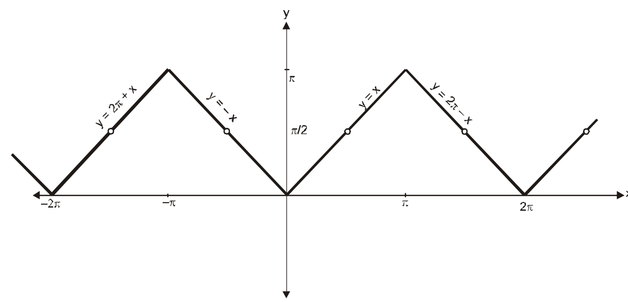
(iv) $\text{cosec}^{-1}(\text{cosec } x)$ is similar to $\sin^{-1}(\sin x)$

Graph of $y = \text{cosec}^{-1}(\text{cosec } x)$



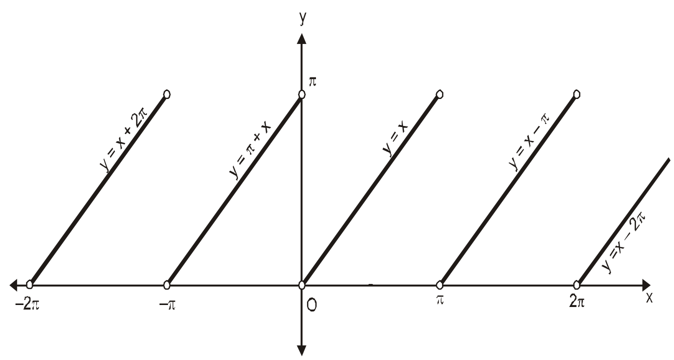
(v) $\sec^{-1}(\sec x)$ is similar to $\cos^{-1}(\cos x)$

Graph of $y = \sec^{-1}(\sec x)$



(vii) $\cot^{-1}(\cot x) = -n\pi + x$, $x \in (n\pi, (n+1)\pi)$, $n \in Z$

Graph of $y = \cot^{-1}(\cot x)$



Remark : $\sin(\sin^{-1}x)$, $\cos(\cos^{-1}x)$, ... $\cot(\cot^{-1}x)$ are aperiodic (non periodic) functions where as $\sin^{-1}(\sin x)$, ..., $\cot^{-1}(\cot x)$ are periodic functions.

Property 4 : "1/x"

(i) $\text{cosec}^{-1}(x) = \sin^{-1}(1/x)$, $|x| \geq 1$

Proof : Let $\text{cosec}^{-1} x = \theta$

$\Rightarrow$ $1/x = \sin\theta$

$\Rightarrow$ $\sin^{-1}(1/x) = \sin^{-1}(\sin\theta)$

$= \theta$ (as $\theta \in [-\pi/2, \pi/2] - \{0\}$)

$= \text{cosec}^{-1}x$

(ii) $\sec^{-1} x = \cos^{-1}(1/x)$, $|x| \geq 1$

(iii) $$\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$$

Property 5 : "π/2"

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, -1 \le x \le 1$

Proof : Let $A = \sin^{-1}x$ and $B = \cos^{-1}x$

$\Rightarrow$ $\sin A = x$ and $\cos B = x$

$\Rightarrow$ $\sin A = \cos B$

$\Rightarrow$ $\sin A = \sin(\pi/2 - B)$

$\Rightarrow$ $A = \pi/2 - B$, because A and $\pi/2 - B \in [-\pi/2, \pi/2]$

$\Rightarrow$ $A + B = \pi/2$.

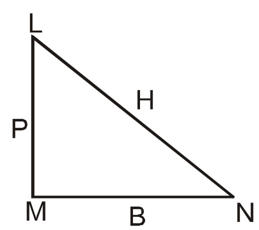
Similarly, we can prove

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \ x \in R$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}, \ |x| \ge 1$

Property- 6 Conversion Property

Let $\sin^{-1} x = y$



$\Rightarrow$ $x = \sin y$

$\Rightarrow$ $\operatorname{cosec} y = \left(\frac{1}{x}\right)$

$\Rightarrow$ $y = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow$ $\sin^{-1}x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$. Hence

(i) $\sin^{-1}x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$ & $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$

Similarly the following results can be obtained

(ii) $\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right)$ & $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$

(iii) $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$ & $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

Solved Examples

Ex.3 $\cos^{-1} \cos\left(\frac{7\pi}{6}\right) =$

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$

(C) $\frac{\pi}{6}$ (D) None of these

Sol. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \ne \frac{7\pi}{6}$

[Because $\frac{7\pi}{6}$ does not lie between 0 and π]

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right]$

$\left(\because \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right)$

[$\because$ $\cos(2\pi - \theta) = \cos\theta$]

$= \frac{5\pi}{6}$ **Ans.[B]**

Ex.4 $\sin\{\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\} =$

(A) 0 (B) –1

(C) 2 (D) 1

Sol. $\sin\{\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\}$

$= \sin\left(\frac{\pi}{2}\right) \left(\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right) = 1$ **Ans.[D]**

Ex.5 $\sin^{-1}\left(\frac{2}{5}\right) =$

(A) $\cos^{-1}\left(\frac{3}{5}\right)$ (B) $\tan^{-1}\left(\frac{3}{5}\right)$

(C) $\operatorname{cosec}^{-1}\left(\frac{5}{2}\right)$ (D) None of these

Sol. We know that $\sin^{-1}x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow$ $\sin^{-1}\left(\frac{2}{5}\right) = \operatorname{cosec}^{-1}\left(\frac{5}{2}\right)$ **Ans. [C]**

Ex.6 Find the value of $\operatorname{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\}$.

Sol. $\because$ $\cot(\cot^{-1}x) = x, \ \forall \ x \in R$

$\therefore$ $\cot\left(\cot^{-1}\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

$\operatorname{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\} = \operatorname{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}$.

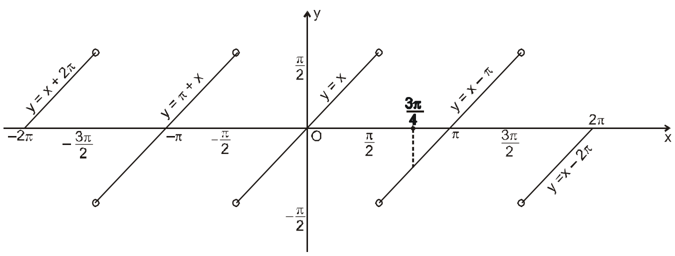
Ex.7 Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Sol. $\because \quad \tan^{-1}(\tan x) = x$

if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ As $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$ $\quad \therefore \frac{3\pi}{4} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

graph of $y = \tan^{-1}(\tan x)$ is as :



$\because$ from the graph we can see that if $\frac{\pi}{2} < x < \frac{3\pi}{2}$,

then $\tan^{-1}(\tan x) = x - \pi$

$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4} - \pi = -\frac{\pi}{4}$

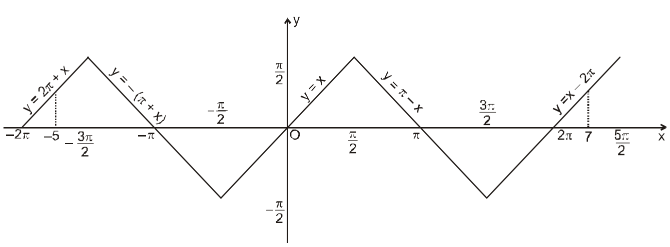
Ex.8 Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Sol. Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \neq 7$ $\quad$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\because \ 2\pi < 7 < \frac{5\pi}{2}$

graph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \le x \le \frac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as : $y = x - 2\pi$

$\therefore \quad \sin^{-1}(\sin 7) = 7 - 2\pi$

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because \quad -2\pi < -5 < -\frac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that

$\sin^{-1}(\sin(-5)) = 2\pi + (-5) = 2\pi - 5$

Ex.9 Find the value of $\cos^{-1}\{\sin(-5)\}$

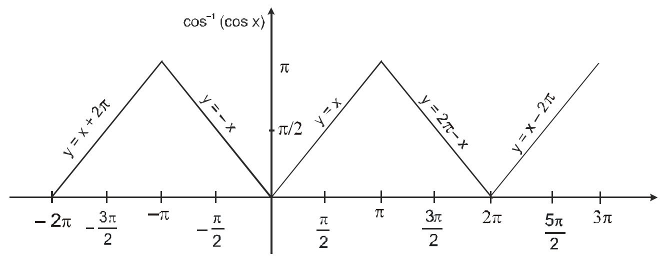
Sol. Let $y = \cos^{-1}\{\sin(-5)\} \quad = \cos^{-1}(-\sin 5)$

$= \pi - \cos^{-1}(\sin 5)$ $\quad (\cos^{-1}(-x) = \pi - \cos^{-1}x, |x| \le 1)$

$= \pi - \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\}$ ....(i)

Note that : $-2\pi < \left(\frac{\pi}{2} - 5\right) < -\pi$

graph of $\cos^{-1}(\cos x)$ is as :



From the graph we can see that if $-2\pi \le x \le -\pi$, then $\cos^{-1}(\cos x) = x + 2\pi$

$\therefore$ from the graph $\cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 5\right)\right\} = \left(\frac{\pi}{2} - 5\right)$ $+ 2\pi = \left(\frac{5\pi}{2} - 5\right)$ $\quad \therefore$ from (i), we get

$\therefore \quad y = \pi - \left(\frac{5\pi}{2} - 5\right) \quad \Rightarrow \quad y = 5 - \frac{3\pi}{2}$.

Solved Examples

Ex.10 Find the value of $\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Sol. Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ....(i)

$\because \quad \cot^{-1}(-x) = \pi - \cot^{-1}x, \ x \in R$

(i) can be written as

$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$ $\quad y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$

$\because \quad \cot^{-1}x = \tan^{-1}\frac{1}{x}$ $\quad$ if $\quad x > 0$

$\therefore \quad y = -\tan\left(\tan^{-1}\frac{3}{2}\right) \quad \Rightarrow \quad y = -\frac{3}{2}$

Ex.11 Find the value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Sol. $\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$

Ex.12 Find the value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Sol. Let $y = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ ....(i)

Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$ and $\cos\theta = \frac{\sqrt{5}}{3}$

$\therefore$ (i) becomes $y = \tan\left(\frac{\theta}{2}\right)$ ....(ii)

$\because \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}} = \frac{3-\sqrt{5}}{3+\sqrt{5}}$

$= \frac{(3-\sqrt{5})^2}{4}$

$\tan\frac{\theta}{2} = \pm\left(\frac{3-\sqrt{5}}{2}\right)$ ....(iii)

$\frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right) \Rightarrow \tan\frac{\theta}{2} > 0$

$\therefore$ from (iii), we get $y = \tan\frac{\theta}{2} = \left(\frac{3-\sqrt{5}}{2}\right)$

Ex.13 Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

Sol. $\cos\left(2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right)$

$= \cos\left(\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5} + \cos^{-1}\frac{1}{5}\right)$

$= \cos\left(\frac{\pi}{2} + \cos^{-1}\frac{1}{5}\right) = -\sin\left(\cos^{-1}\left(\frac{1}{5}\right)\right)$ ....(i)

$= -\sqrt{1-\left(\frac{1}{5}\right)^2} = -\frac{2\sqrt{6}}{5}$.

IDENTITIES FOR SUM & DIFFERENCE OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$,

if $x > 0, y > 0$ and $xy < 1$

(ii) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$,

if $x > 0, y > 0$ and $xy > 1$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(iv) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z$

$= \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

(vi) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}]$,

if $x, y \geq 0$ and $x^2 + y^2 \leq 1$.

(vii) $\sin^{-1}x \pm \sin^{-1}y$

$= \pi - \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}]$,

if $x, y \geq 0$ and $x^2 + y^2 > 1$.

(viii) $\cos^{-1}x \pm \cos^{-1}y$

$= \cos^{-1}[xy \mp \sqrt{1-x^2}\ \sqrt{1-y^2}]$,

if $x, y > 0$ and $x^2 + y^2 \leq 1$.

(ix) $\cos^{-1}x \pm \cos^{-1}y$

$= \pi - \cos^{-1}[xy \mp \sqrt{1-x^2}\ \sqrt{1-y^2}]$,

if $x, y > 0$ and $x^2 + y^2 > 1$.

(x) $\cot^{-1}x \pm \cot^{-1}y = \cot^{-1}\left[\frac{xy \mp 1}{y \pm x}\right]$

Solved Examples

Ex.14 $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) =$

(A) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$

(C) $\tan^{-1}\left(\frac{3}{5}\right)$ (D) $\tan^{-1}\left(\frac{1}{2}\right)$

Sol. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\frac{2}{9}}\right)$

$= \tan^{-1}\left(\frac{17}{34}\right) = \tan^{-1}\left(\frac{1}{2}\right)$ **Ans.[D]**

Ex.15 The value of

$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to-

(A) $\frac{\pi}{4}$ (B) $\frac{5\pi}{12}$

(C) $\frac{3\pi}{4}$ (D) $\frac{13\pi}{12}$

Sol. $\tan^{-1}(1) + \cot^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$ **Ans.[C]**

Ex.16 The number of solution of the equation

$\tan^{-1}(x - 1) + \tan^{-1}x + \tan^{-1}(x + 1) = \tan^{-1} 3x$ is-

(A) 1 (B) 2

(C) 3 (D) 4

Sol. The given equation can be written as

$\tan^{-1}(x - 1) + \tan^{-1}(x + 1) = \tan^{-1} 3x - \tan^{-1}x$

$\Rightarrow \tan^{-1} \frac{x-1+x+1}{1-(x-1)(x+1)} = \tan^{-1} \frac{3x-x}{1+3x^2}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2} \Rightarrow x + 3x^3 = 2x - x^3$

$\Rightarrow 4x^3 - x = 0 \quad \Rightarrow x(4x^2 - 1) = 0$

$\Rightarrow x = 0, x = \pm \frac{1}{2}$ **Ans.[C]**

INVERSE TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$, if $-1 \le x \le 1$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$, if $-1 \le x \le 1$

(iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(iv) $3 \sin^{-1}x = \sin^{-1}(3x - 4x^3)$

(v) $3 \cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(vi) $3 \tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Solved Examples

Ex.17 $\cos^{-1}\left(\frac{15}{17}\right) + 2 \tan^{-1}\left(\frac{1}{5}\right) =$

(A) $\frac{\pi}{2}$ (B) $\cos^{-1}\left(\frac{140}{221}\right)$

(C) $\frac{\pi}{4}$ (D) None of these

Sol. $\cos^{-1}\left(\frac{15}{17}\right) + 2 \tan^{-1}\left(\frac{1}{5}\right)$

$= \cos^{-1}\left(\frac{15}{17}\right) + \cos^{-1}\left(\frac{1-\frac{1}{25}}{1+\frac{1}{25}}\right)$

$= \cos^{-1}\left(\frac{15}{17}\right) + \cos^{-1}\left(\frac{12}{13}\right)$

$= \cos^{-1}\left(\frac{15}{17}\times\frac{12}{13} - \sqrt{1-\left(\frac{15}{17}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$

$= \cos^{-1}\left(\frac{140}{221}\right)$ **Ans.[B]**

Ex.18 The value of

$\sin(2 \tan^{-1} \frac{1}{3}) + \cos (\tan^{-1} 2\sqrt{2})$ is–

(A) $\frac{6}{15}$ (B) $\frac{7}{15}$

(C) $\frac{14}{15}$ (D) None of these

Sol. Let $\tan^{-1} \frac{1}{3} = \alpha$ and $\tan^{-1} 2\sqrt{2} = \beta$. Then $\tan\alpha = \frac{1}{3}$ and $\tan\beta = 2\sqrt{2}$, so that

$\sin (2 \tan^{-1} \frac{1}{3}) + \cos (\tan^{-1} 2\sqrt{2})$

$= \sin 2\alpha + \cos\beta = \frac{2\tan\alpha}{1+\tan^2\alpha} + \frac{1}{\sqrt{1+\tan^2\beta}}$

$= \frac{2.\frac{1}{3}}{1+\frac{1}{9}} + \frac{1}{\sqrt{1+8}} = \frac{2}{3}.\frac{9}{10} + \frac{1}{3}$

$= \frac{3}{5} + \frac{1}{3} = \frac{14}{15}$ **Ans.[C]**

MISCELLENEOUS RESULTS

(i) $\tan^{-1}\left[\frac{x}{\sqrt{a^2-x^2}}\right] = \sin^{-1}\left(\frac{x}{a}\right)$

(ii) $\tan^{-1}\left[\frac{3a^2x - x^3}{a(a^2-3x^2)}\right] = 3 \tan^{-1}\left(\frac{x}{a}\right)$

(iii) $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

(iv) $\sin^{-1}(x) = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

(v) $\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(vi) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\sqrt{1+x^2}\right)$

$= \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$