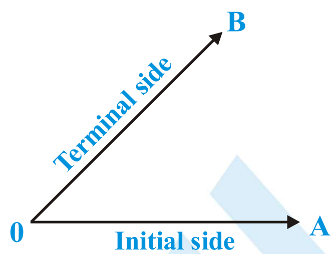
# TRIGONOMETRIC RATIO & IDENTITIES

## INTRODUCTION

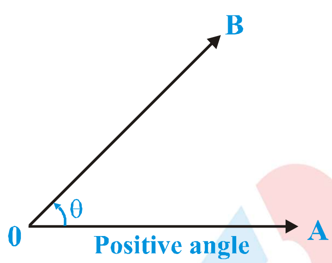
The word "Trigonometry" is derived from two Greek words : (a) Trigonon and, (b) Metron. The word trigonon means a triangle and the word metron means a measure. Hence, trigonometry means the science of measuring triangles. In broader sense it is that branch of mathematics which deals with the measurement of the sides and the angles of a triangle and the problems allied with angles.

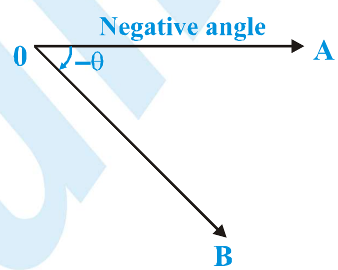
## ANGLES

ANGLE Consider a ray $\vec{OA}$ . If this ray rotates about its end point O and takes the position OB, then we say that the angle ∠AOB has been generated.




Thus, an angle is considered as the figure obtained by rotating a given ray about its end-point. The end point O about which the ray rotates is called the vertex of the angle.




## SYSTEM OF MEASUREMENT OF ANGLES

### Sexagesimal System

In this system a right angle is divided into 90 equal parts, called degrees. The symbol 1° is used to denote one degree.

Thus, 1 right angle = 90 degree (= 90°)

1° = 60 minutes (= 60 ')

1' = 60 seconds (= 60")

### Centesimal System

In this system a right angle is divided into 100 equal parts, called grades ; each grade is subdivided into 100 minutes, and each minute into 100 seconds.

Thus, 1 right angle = 100 grades (= $100^g$)

1 grade = 100 minutes (=100 ')

1 minute = 100 seconds (100")

### Circular System

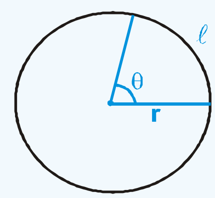
Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

**(i)** Relation between the three systems : $\frac{D}{90} = \frac{G}{100} = \frac{R}{\pi/2}$

**(ii)** If $\theta$ is the angle subtended at the centre of a circle of radius 'r',

by an arc of length '●' then $\frac{\lambda}{r} = \theta$ .

Here ●, r are in the same units and $\theta$ is always in radians.




$\theta \in [0, \pi/2]$

**Ex.** Express 1.2 rad in degree measure.

**Sol.** $(1.2)^R = 1.2 \times \left(\frac{180}{\pi}\right)^o = 1.2 \times \frac{180 \times 7}{22}$ $\left[Q\ \pi = \frac{22}{7}(\text{approx})\right]$

$= 68.7272 = 68°\ (0.7272 \times 60)' = 68°(43.63)' = 60°43'\ (0.63 \times 60)'' = 68°(43'.37.8'')$

**Ex.** Find in degree the angle subtended at the center of a circle of diameter 50 cm by an arc of length 11 cm.

**Sol.** Here, r = 25 cm and s = 11 cm. Therefore,

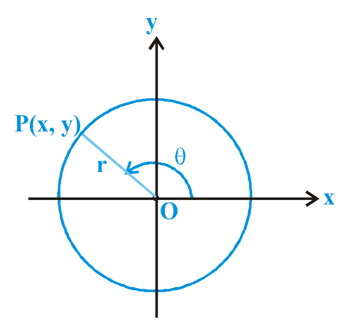
$$\theta = \left(\frac{s}{r}\right)^R \text{ or } \theta = \left(\frac{11}{25}\right)^R = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^o$$

$$= \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^o$$

$$= \left(\frac{126}{5}\right)^o = \left(25\frac{1}{5}\right)^o = 25\left(\frac{1}{5} \times 60\right)' = 25°12'$$

## TRIGONOMETERY RATIOS (T -RATIOS)

By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle $\theta$ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms




of r and the coordinates x and y.

$\sin\theta = y/r$,

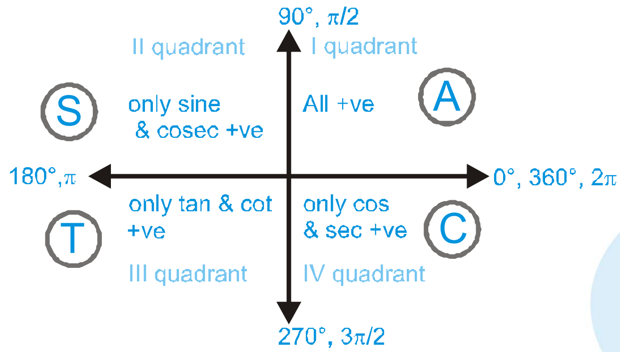
$\cos\theta = x/r$

$\tan\theta = y/x$,

(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

## SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS




## T-RATIOS OF SOME STANDARD ANGLES

| Angles → T-ratio ↓ | 0° (0) | 30° (π/6) | 45° (π/4) | 60° (π/3) | 90° (π/2) | 180° (π) | 270° (3π/2) |
|---|---|---|---|---|---|---|---|
| sinθ | 0 | 1/2 | 1/√2 | √3/2 | 1 | 0 | -1 |
| cosθ | 1 | √3/2 | 1/√2 | 1/2 | 0 | -1 | 0 |
| tanθ | 0 | 1/√3 | 1 | √3 | N.D. | 0 | N.D. |
| cotθ | N.D. | √3 | 1 | 1/√3 | 0 | N.D. | 0 |
| secθ | 1 | 2/√3 | √2 | 2 | N.D. | -1 | N.D. |
| cosecθ | N.D. | 2 | √2 | 2/√3 | 1 | N.D. | -1 |

**(i)** $\sin n\pi = 0$ ; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

**(ii)** $\sin(2n+1)\frac{\pi}{2} = (-1)^n$; $\cos(2n+1)\frac{\pi}{2} = 0$ where $n \in I$

## TRIGONOMETERY RATIOS OF ALLIED ANGLES

**Definition :** "Two angles are said to be allied when their sum or difference is either zero or a multiple of 90° ".

The angles $-\theta$, $90° \pm\theta$, $180° \pm\theta$, $360° \pm\theta$ etc are angles allied to the angle θ if θ is measured in degrees.

However, if θ is measured in radians, then the angles allied to θ are $-\theta$, $\frac{\pi}{2} \pm\theta$, $\pi \pm \theta$, $2\pi \pm \theta$ etc.

$\sin(2n\pi + \theta) = \sin\theta$, | $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

$\sin(-\theta) = -\sin\theta$ | $\cos(-\theta) = \cos\theta$

$\sin(90° - \theta) = \cos\theta$ | $\cos(90° - \theta) = \sin\theta$

| | |
|---|---|
| $\sin(90° + \theta) = \cos\theta$ | $\cos(90° + \theta) = -\sin\theta$ |
| $\sin(180° - \theta) = \sin\theta$ | $\cos(180° - \theta) = -\cos\theta$ |
| $\sin(180° + \theta) = -\sin\theta$ | $\cos(180° + \theta) = -\cos\theta$ |
| $\sin(270° - \theta) = -\cos\theta$ | $\cos(270° - \theta) = -\sin\theta$ |
| $\sin(270° + \theta) = -\cos\theta$ | $\cos(270° + \theta) = \sin\theta$ |
| $\sin(360° - \theta) = -\sin\theta$ | $\cos(360° - \theta) = \cos\theta$ |
| $\sin(360° + \theta) = \sin\theta$ | $\cos(360° + \theta) = \cos\theta$ |

**Ex.** Show that $\tan 1° \tan 2° \ldots \tan 89° = 1$

**Sol.** L.H.S $= (\tan 1° \tan 89°)(\tan 2° \tan 88°) \ldots (\tan 44° \tan 46°) \tan 45°$

$= [\tan 1° \tan(90° - 1°)][\tan 2° \tan(90° - 2°)] \ldots [\tan 44° \tan(90° - 44°)] \tan 45°$

$= (\tan 1° \cot 1°)(\tan 2° \cot 2°) \ldots (\tan 44° \cot 44°) \tan 45°$

$= 1$ [∵ $\tan\theta \cot\theta = 1$ and $\tan 45° = 1$]

= R.H.S

## BASIC TRIGONOMETRIC IDENTITIES

**(1)** $\sin\theta . \operatorname{cosec}\theta = 1$ **(2)** $\cos\theta . \sec\theta = 1$ **(3)** $\tan\theta . \cot\theta = 1$

**(4)** $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ **(5)** $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

**(6)** $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

**(7)** $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

**(8)** $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$ **(9)** $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$

**(10)** $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

**Ex.** If $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$, $\tan\theta$, and $\sec\theta$ for $\theta \in [0, \pi/2]$

**Sol.** Given, $\sec\theta + \tan\theta = \dfrac{3}{2}$ ....(i)

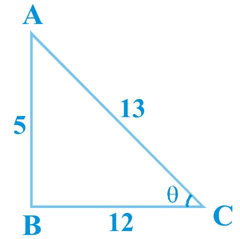
Now, $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta} = \dfrac{2}{3}$ ....(ii)

Adding eqs. (i) and (ii), we get $2\sec\theta = \dfrac{3}{2} + \dfrac{2}{3} = \dfrac{13}{6}$

∴ $\sec\theta = \dfrac{13}{12}$

∴ $\tan\theta = \dfrac{5}{12}$

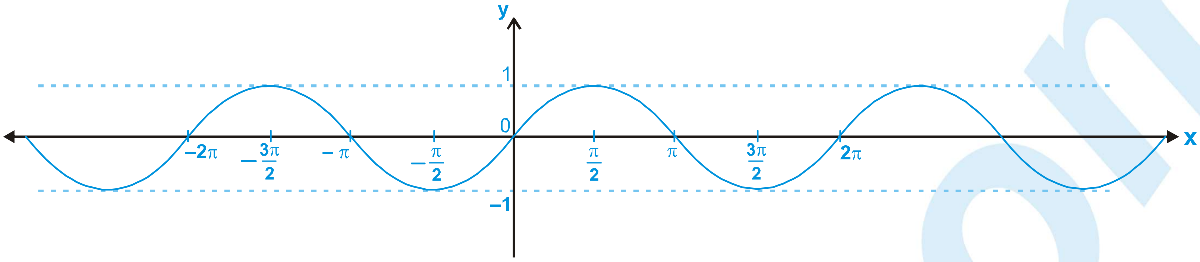
and $\sin\theta = \dfrac{5}{13}$



## GRAPH OF TRIGONOMETRIC FUNCTIONS

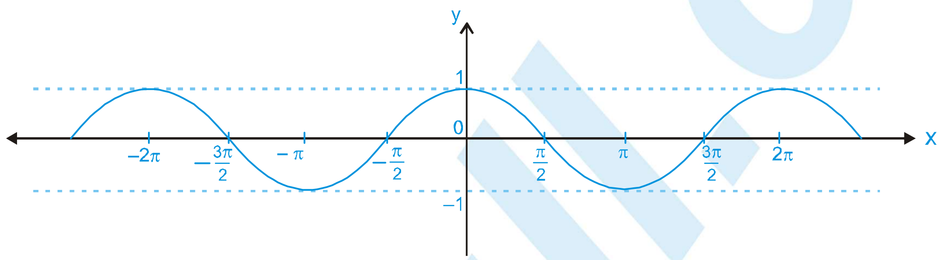
**(i)** **y = sin x**

Domain : $x \in R$
Range : $y \in [-1, 1]$



**(ii)** **y = cos x**

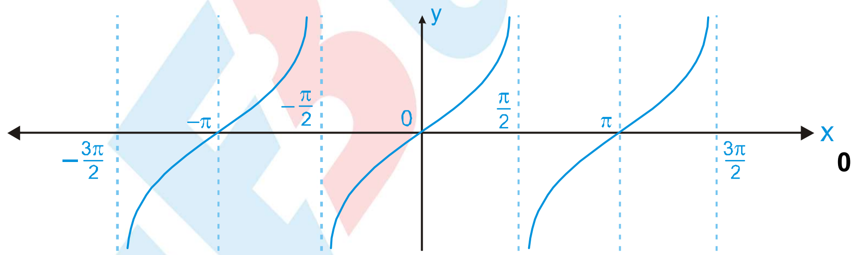
Domain : $x \in R$
Range : $y \in [-1, 1]$



**(iii)** **y = tan x**

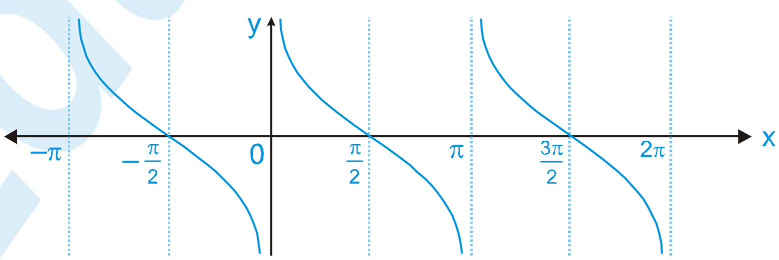
Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}, \quad n \in I$

Range : $y \in R$



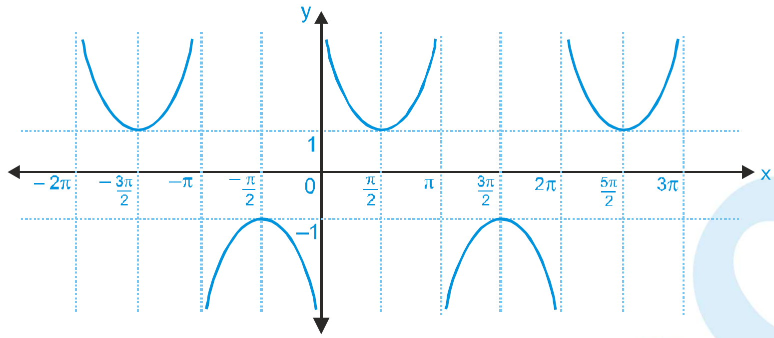
**(iv)** **y = cot x**

Domain : $x \in R - \{n\pi\}, \quad n \in I$
Range : $y \in R$



**(v)** **y = cosec x**

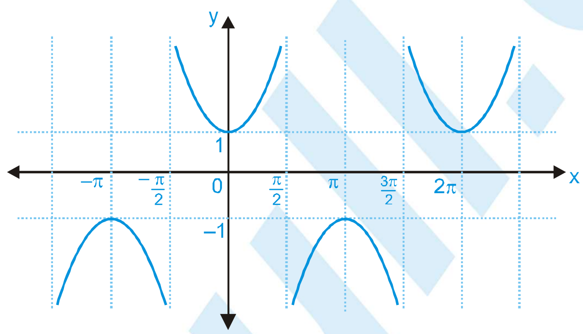
Domain : $x \in R - \{n\pi\}, \quad n \in I$
Range : $y \in (-\infty, -1] \cup [1, \infty)$




**(vi)** **y = sec x**

Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}, \quad n \in I$

Range : $y \in (-\infty, -1] \cup [1, \infty)$




**Ex.** Show that the equation $\sin\theta = x + \frac{1}{x}$ is impossible if x is real.

**Sol.** Given, $\sin\theta = x + \frac{1}{x}$

$\therefore \quad \sin^2\theta = x^2 + \frac{1}{x^2} + 2x\frac{1}{x} = x^2 + \frac{1}{x^2} + 2 = \left(x - \frac{1}{x}\right)^2 + 4 \geq 4$

which is not possible since $\sin^2\theta \leq 1$.

## TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES

**(i)** $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

**(ii)** $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

**(iii)** $\cos(A+B) = \cos A \cos B - \sin A \sin B$

**(iv)** $\cos(A-B) = \cos A \cos B + \sin A \sin B$

**(v)** $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

**(vi)** $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

**(vii)** $\cot(A+B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$

**(viii)** $\cot(A-B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$

### Other useful results

**(i)** $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

**(ii)** $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B) = \cos^2 B - \sin^2 A$

**(iii)** $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

**Ex.** Prove that $\tan 70° = \cot 70° + 2\cot 40°$.

**Sol.** L.H.S. $= \tan 70° = \tan(20° + 50°) = \dfrac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$

or $\tan 70° - \tan 20° \tan 50° \tan 70° = \tan 20° + \tan 50°$

or $\tan 70° = \tan 70° \tan 50° \tan 20° + \tan 20° + \tan 50° = 2\tan 50° + \tan 20°$

$= \cot 70° + 2\cot 40°$ = R.H.S.

## TRANSFORMATION FORMULA

### Formulae to Transform the Production Sum or Difference

**(i)** $2\sin A \cos B = \sin(A+B) + \sin(A-B)$.

**(ii)** $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.

**(iii)** $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

**(iv)** $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

**Ex.** If $\sin 2A = \lambda \sin 2B$, then prove that $\dfrac{\tan(A+B)}{\tan(A-B)} = \dfrac{\lambda+1}{\lambda-1}$

**Sol.** Given $\sin 2A = \lambda \sin 2B$

$\Rightarrow \dfrac{\sin 2A}{\sin 2B} = \dfrac{\lambda}{1}$

Applying componendo & dividendo,

$$\frac{\sin 2A + \sin 2B}{\sin 2B - \sin 2A} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin\{-(A-B)\}} = \frac{\lambda+1}{1-\lambda} \qquad \Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\times -\sin(A-B)} = \frac{\lambda+1}{-(\lambda-1)}$$

$$\Rightarrow \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)} = \frac{\lambda+1}{\lambda-1} \qquad \Rightarrow \tan(A+B)\cot(A-B) = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$$

### Formulae to Transform Sum or Difference into Product

**(i)** $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$ **(ii)** $\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

**(iii)** $\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$ **(iv)** $\cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

**Ex.** Prove that $\cos 18° - \sin 18° = \sqrt{2}\sin 27°$.

**Sol.** LHS $= \cos 18° - \sin 18° = \cos 18° - \sin(90° - 72°) = \cos 18° - \cos 72°$

$$= 2\sin\frac{18°+72°}{2}\sin\frac{72°-18°}{2}$$

$$= 2\sin 45°\sin 27° = 2\frac{1}{\sqrt{2}}\sin 27° = \sqrt{2}\sin 27°$$

**Ex.** Show that sin12°.sin48°.sin54° = 1/8

**Sol.** L.H.S. $= \frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36°\sin 54° - \frac{1}{2}\sin 54°\right]$

$$= \frac{1}{4}[2\cos 36°\sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$$

$$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18°\cos 36°]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18°\cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36°\cos 36°}{\cos 18°}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36°\cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$$

## MULTIPLE AND SUB-MULTIPLE ANGLES

**(a)** $\sin 2A = 2\sin A\cos A$, **Also** $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

**(b)** $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

**Also** $2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$.

**(c)** $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$, **Also** $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$

**(d)** $\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$, $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

**(e)** $\sin 3A = 3\sin A - 4\sin^3 A$

**(f)** $\cos 3A = 4\cos^3 A - 3\cos A$

**(g)** $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

**Ex.** Prove that $\frac{\cos 2\theta}{1+\sin 2\theta} = \tan(\pi/4 - \theta)$

**Sol.** LHS $= \frac{\cos 2\theta}{1+\sin 2\theta}$

$$= \frac{\sin\left(\frac{\pi}{2}-2\theta\right)}{1+\cos\left(\frac{\pi}{2}-2\theta\right)}$$

$$= \frac{2\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4}-\theta\right)}$$

$$= \tan\left(\frac{\pi}{4}-\theta\right) = \text{RHS}$$

**Ex.** Prove that : $\frac{2\cos 2A+1}{2\cos 2A-1} = \tan(60°+A)\tan(60°-A)$.

**Sol.** R.H.S. $= \tan(60°+A)\tan(60°-A)$

$$= \left(\frac{\tan 60°+\tan A}{1-\tan 60°\tan A}\right)\left(\frac{\tan 60°-\tan A}{1+\tan 60°\tan A}\right) = \left(\frac{\sqrt{3}+\tan A}{1-\sqrt{3}\tan A}\right)\left(\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}\right)$$

$$= \frac{3-\tan^2 A}{1-3\tan^2 A} = \frac{3-\frac{\sin^2 A}{\cos^2 A}}{1-3\frac{\sin^2 A}{\cos^2 A}} = \frac{3\cos^2 A-\sin^2 A}{\cos^2 A-3\sin^2 A} = \frac{2\cos^2 A+\cos^2 A-2\sin^2 A+\sin^2 A}{2\cos^2 A-2\sin^2 A-\sin^2 A-\cos^2 A}$$

$$= \frac{2(\cos^2 A-\sin^2 A)+\cos^2 A+\sin^2 A}{2(\cos^2 A-\sin^2 A)-(\sin^2 A+\cos^2 A)} = \frac{2\cos 2A+1}{2\cos 2A-1} = \text{L.H.S.}$$

**Ex.** Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta,\ 0<\theta<\pi/16$.

**Sol.** LHS $= \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}} \qquad \left[\because 1+\cos 8\theta = 2\cos^2\frac{8\theta}{2}\right]$$

$$= \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4\theta)}}}$$

$$= \sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$= \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} \qquad [\because 1+\cos 4\theta = 2\cos^2 2\theta]$$

$$= \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)} = 2\cos\theta = \text{RHS}.$$

## IMPORTANT TRIGONOMETRIC RATIOS OF STANDARD ANGLES

$\sin 15°$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\frac{5\pi}{12}$ ;

$\cos 15°$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\frac{5\pi}{12}$ ;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$ ; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

$\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72°$

$\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54°$

## CONDITIONAL IDENTITIES

If $A + B + C = \pi$ then

**(i)** $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

**(ii)** $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

**(iii)** $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

**(iv)** $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

**(v)** $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

**(vi)** $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

**(vii)** $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2} \cdot \cot\frac{B}{2} \cdot \cot\frac{C}{2}$

**(viii)** $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

**(ix)** $A + B + C = \frac{\pi}{2}$ then $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$

**Ex.** If $A + B + C = 180°$, Prove that, $\sin^2A + \sin^2B + \sin^2C = 2 + 2\cos A \cos B \cos C$.

**Sol.** Let $S = \sin^2A + \sin^2B + \sin^2C$

so that $2S = 2\sin^2A + 1 - \cos 2B + 1 - \cos 2C$

$= 2\sin^2A + 2 - 2\cos(B + C)\cos(B - C)$

$= 2 - 2\cos^2A + 2 - 2\cos(B + C)\cos(B - C)$

$\therefore$ $S = 2 + \cos A\,[\cos(B - C) + \cos(B + C)]$

since $\cos A = -\cos(B+C)$

$\therefore$ $S = 2 + 2\cos A \cos B \cos C$

## IMPORTANT RESULTS

**(i)** $\sin\theta \sin(60° - \theta)\sin(60° + \theta) = \frac{1}{4}\sin 3\theta$

**(ii)** $\cos\theta . \cos(60° - \theta)\cos(60° + \theta) = \frac{1}{4}\cos 3\theta$

**(iii)** $\tan\theta \tan(60° - \theta)\tan(60° + \theta) = \tan 3\theta$

**(iv)** $\cot\theta \cot(60° - \theta)\cot(60° + \theta) = \cot 3\theta$

**(v)** **(a)** $\sin^2\theta + \sin^2(60° + \theta) + \sin^2(60° - \theta) = \frac{3}{2}$

**(b)** $\cos^2\theta + \cos^2(60° + \theta) + \cos^2(60° - \theta) = \frac{3}{2}$

**(vi)** **(a)** If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi, n \in I$

**(b)** If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

**(vii)** $\cos\theta \cos 2\theta \cos 4\theta .... \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n \sin\theta}$

**(viii)** **(a)** $\cot A - \tan A = 2\cot 2A$

**(b)** $\cot A + \tan A = 2\text{cosec}2A$

**(ix)** $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ... \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

**(x)** $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + .... + \cos(\alpha + \overline{n-1}\,\beta) = \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

**Ex.** Find the value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$.

**Sol.** $S = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

$$= \frac{\sin\left(3\frac{\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}\cos\left(\frac{\pi}{7} + \frac{3\pi}{7}\right)$$

$$= \frac{2\sin\left(\frac{3\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = \frac{\sin\left(\frac{7\pi}{7}\right) - \sin\left(\frac{\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = -\frac{1}{2}$$

**Ex.** Prove that $\sin 10°\sin 30°\sin 50°\sin 70° = 1/16$.

**Sol.** $\sin 10°\sin 30°\sin 50°\sin 70° = \sin 10°\sin(60° - 10°)\sin(60° + 10°)\sin 30°$

$$= \frac{1}{4}\sin(3 \times 10°)\sin 30° = \frac{1}{4}\sin^2 30° = \frac{1}{16}$$

## MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS

**(i)** $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

**(ii)** Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $a, b > 0$

**(iii)** $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where $\alpha$ and $\beta$ are known angles.

**(iv)** If $\alpha, \beta, \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then

- **(i)** Maximum value of the expression $\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha\sin\beta$ or $\sin\alpha + \sin\beta$ occurs when $\alpha = \beta = \sigma/2$
- **(ii)** Minimum value of $\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\text{cosec}\,\alpha + \text{cosec}\,\beta$ occurs when $\alpha = \beta = \sigma/2$

**(v)** If A, B, C are the angles of a triangle then maximum value of
$\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60°$

**(vi)** In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

**Ex.** Find the maximum value of $\sqrt{3}\sin x + \cos x$ and x for which a maximum value occurs.

**Sol.** $\sqrt{3}\sin x + \cos x = 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\sin x\right) = 2\sin\left(x + \frac{\pi}{6}\right)$

which is maximum when $x + \pi/6 = \pi/2$ or $x = 60°$ and has a maximum value 2.

**Ex.** Find the maximum and minimum values of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

**Sol.** $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$

$$= \frac{1+\cos 2\theta}{2} - 3\sin 2\theta + 3\frac{(1-\cos 2\theta)}{2} + 2$$

$$= 4 - \cos 2\theta - 3\sin 2\theta$$

Now, $-\cos 2\theta - 3\sin 2\theta \in \left[-\sqrt{10}, \sqrt{10}\right]$

$\Rightarrow$ $4 - \cos 2\theta - 3\sin 2\theta \in \left[4-\sqrt{10}, 4+\sqrt{10}\right]$

## 1. Relation Between System of Measurement of Angles

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

$$1 \text{ Radian} = \frac{180}{\pi} \text{ degree} \approx 57o\ 17'15'' \text{ (approximately)}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radian} \approx 0.0175 \text{ radian}$$

## 2. Basic Trigonometric Identities

(a) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

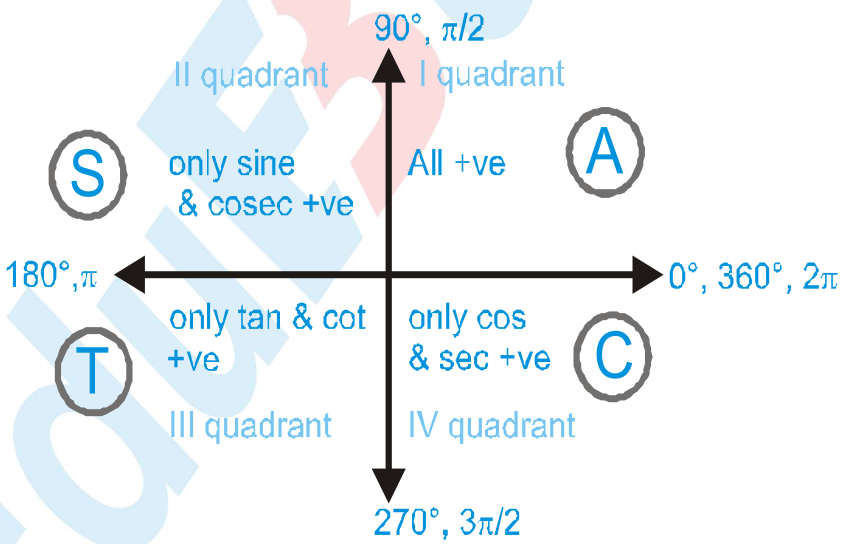
(b) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(c) If $\sec\theta + \tan\theta = k \Rightarrow \sec\theta - \tan\theta = \frac{1}{k} \Rightarrow 2\sec\theta = k + \frac{1}{k}$

(d) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $\text{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \text{cosec}^2\theta - 1$

(e) If $\text{cosec}\theta + \cot\theta = k \Rightarrow \text{cosec}\theta - \cot\theta = \frac{1}{k} \Rightarrow 2\,\text{cosec}\theta = k + \frac{1}{k}$

## 3. Signs of Trigonometric Functions in Different Quadrants



## 4. Trigonometric Functions of Allied Angles

**(a)** $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

**(b)** $\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\sin(90^o - \theta) = \cos\theta$ $\qquad$ $\cos(90^o - \theta) = \sin\theta$

$\sin(90^o + \theta) = \cos\theta$ $\qquad$ $\cos(90^o + \theta) = -\sin\theta$

$\sin(180^o - \theta) = \sin\theta$ $\qquad$ $\cos(180^o - \theta) = -\cos\theta$

$\sin(180^o + \theta) = -\sin\theta$ $\qquad$ $\cos(180^o + \theta) = -\cos\theta$

$\sin(270^o - \theta) = -\cos\theta$ $\qquad$ $\cos(270^o - \theta) = -\sin\theta$

$\sin(270^o + \theta) = -\cos\theta$ $\qquad$ $\cos(270^o + \theta) = \sin\theta$

❖ **(i)** $\sin n\pi = 0$ ; $\cos n\pi = (-1)^o$ ; $\tan n\pi = 0$ where $n \in I$

❖ **(ii)** $\sin(2n+1)\dfrac{\pi}{2} = (-1)^o$ ; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$

## 5. Important Trigonometric Formulae

**(i)** $\sin(A+B) = \sin A\cos B + \cos A\sin B.$

**(ii)** $\sin(A-B) = \sin A\cos B - \cos A\sin B.$

**(iii)** $\sin(A+B) = \sin A\cos B - \cos A\sin B.$

**(iv)** $\sin(A-B) = \sin A\cos B + \cos A\sin B.$

**(v)** $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

**(vi)** $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

**(vii)** $\cot(A+B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$

**(viii)** $\cot(A-B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$

**(ix)** $2\sin A\cos B = \sin(A+B) + \sin(A-B).$

**(x)** $2\cos A\sin B = \sin(A+B) - \sin(A-B).$

**(xi)** $2\cos A\cos B = \cos(A+B) + \cos(A-B).$

**(xii)** $2\sin A\sin B = \cos(A-B) - \cos(A+B).$

**(xiii)** $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

**(xiv)** $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

**(xv)** $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

**(xvi)** $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

**(xvii)** $\sin 2\theta = 2\sin\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

**(xviii)** $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

**(xix)** $1 + \cos 2\theta = 2\cos^2\theta$ or $\cos\theta = \pm\sqrt{\dfrac{1+\cos 2\theta}{2}}$

**(xx)** $1 - \cos 2\theta = 2\sin^2\theta$ or $\sin\theta = \pm\sqrt{\dfrac{1-\cos 2\theta}{2}}$

**(xxi)** $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

**(xxii)** $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

**(xxiii)** $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

**(xxiv)** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

**(xxv)** $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

**(xxvi)** $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

**(xxvii)** $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

**(xxviii)** $\sin(A+B+C)$

$= \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B$

$- \sin A\sin B\sin C$

$= \Sigma\sin A\cos B\cos C - \prod \sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

**(xxix)** $\cos(A+B+C)$

$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C$

$- \cos A\sin B\sin C$

$= \prod \cos A - \Sigma\sin A\sin B\cos C$

$= \cos A\cos B\cos C\,[1 - \tan A\tan B - \tan C - \tan C\tan A]$

**(xxx)** $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A} = \frac{S_1 - S_3}{1 - S_2}$$

**(xxxi)** $\sin\alpha + \sin(\alpha+\beta) + \ldots\ldots + \sin(\alpha + n - 1\beta)$

$$= \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

**(xxxii)** $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

## 6. Values of Some T–Ratios for Angles 18°, 36°, 15°, 22.5°, 67.5° etc.

(a) $\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \sin\frac{\pi}{10}$

(b) $\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ = \cos\frac{\pi}{5}$

(c) $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ = \sin\frac{\pi}{12}$

(d) $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} \sin 75^\circ = \cos\frac{\pi}{12}$

(e) $\tan\frac{\pi}{12} = 2 - \sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot\frac{5\pi}{12}$

(f) $\tan\frac{5\pi}{12} = 2 + \sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot\frac{\pi}{12}$

(g) $\tan(225^\circ) = \sqrt{2} - 1 = \cot(67.5^\circ) = \cot\frac{3\pi}{8} = \tan\frac{\pi}{8}$

(h) $\tan(67.5^\circ) = \sqrt{2} + 1 = \cot(225^\circ)$

## 7. Maximum & Minimum Values of Trigonometric Expressions

(a) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(b) Minimum values of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $a, b > 0$

(c) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where $\alpha$ and $\beta$ are known angles.

(d) Minimum value of $a^2\cos^2\theta + b^2\sec^2\theta$ is either $2ab$ or $a^2 + b^2$, if for some real $\theta$ equation $a\cos\theta = b\sec\theta$ is true or not true $\{a, b > 0\}$

(e) Minimum value of $a^2\sin^2\theta + b^2\operatorname{cosec}^2\theta$ is either $2ab$ or $a^2 + b^2$, if for some real $\theta$ equation $a\sin\theta = b\operatorname{cosec}\theta$ is true or not true $\{a, b, 0\}$

## 8. Important Results

(a) $\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta) = \frac{1}{4}\sin 3\theta$

(b) $\cos\theta\cos(60^\circ-\theta)\cos(60^\circ+\theta) = \frac{1}{4}\cos 3\theta$

(c) $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta) = \tan 3\theta$

(d) $\cot\theta\cot(60^\circ-\theta)\cot(60^\circ+\theta) = \cot 3\theta$

(e) (i) $\sin^2\theta + \sin^2(60^\circ+\theta) + \sin^2(60^\circ-\theta) = \frac{3}{2}$

(ii) $\cos^2\theta + \cos^2(60^\circ+\theta) + \cos^2(60^\circ-\theta) = \frac{3}{2}$

**(f)** **(i)** If tan A + tan B + tan C = tan A tan BN tan C,

then $A + B + C = n\pi, n \in 1$

**(ii)** If tan A tan B + tan B tan C + tan C tan A = 1,

then $A + B + C = (2n+1)\dfrac{\pi}{2}, n \in 1$

**(g)** $\cos\theta \cos 2\theta \cos 4\theta \ldots\ldots \cos(2^{n-1}\theta) = \dfrac{\sin(2^n\theta)}{2^n \sin\theta}$

**(h)** cot A – tan A = 2cot2A

## 9. Conditional Identities

If A + B + C = 180º, then

**(a)** tan A + tan B + tan C = tanA tanB tanC

**(b)** cot A cot B + cot B cot C + cot C cot A = 1

**(c)** $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1$

**(d)** $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

**(e)** sin 2A + sin 2B + sin 2C = 4 sin A sin B sin C

**(f)** cos 2A + cos 2B + cos 2C = –1 – 4 cosA cosB cosC

**(g)** $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

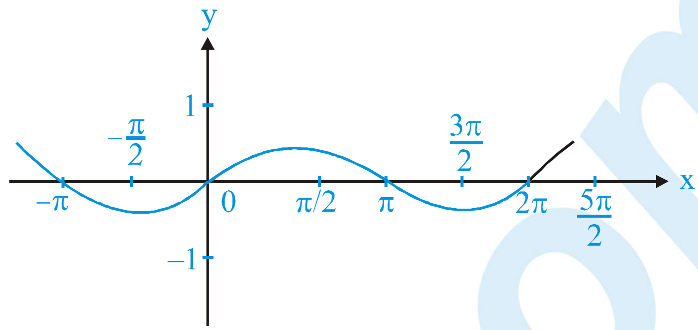
**(h)** $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

## 10. Domains, Ranges and Periodicity of Trigonometric Functions

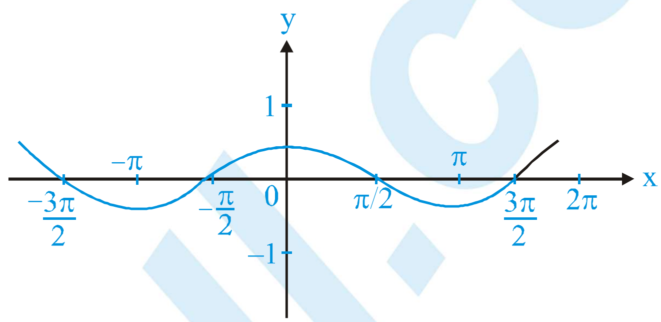
| T-Ratio | Domain | Range | Period |
|---|---|---|---|
| sinx | R | [–1, 1] | 2π |
| cosx | R | [–1, 1] | 2π |
| tanx | R– [(2n + 1)π/2 ; n∈I] | R | π |
| cotx | R– [nπ : ∈I] | R | π |
| secx | R– [(2n + 1)π/2 : n∈I] | (–∞ – 1] ∪ [1,∞) | 2π |
| cosecx | R– {nπ : n∈I} | (∞–, – 1] ∪ [1,∞) | 2π |

## 11. Graph of Trigonometric Functions

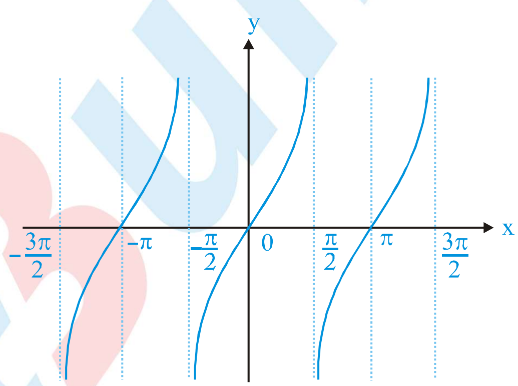
(a) y = sin x



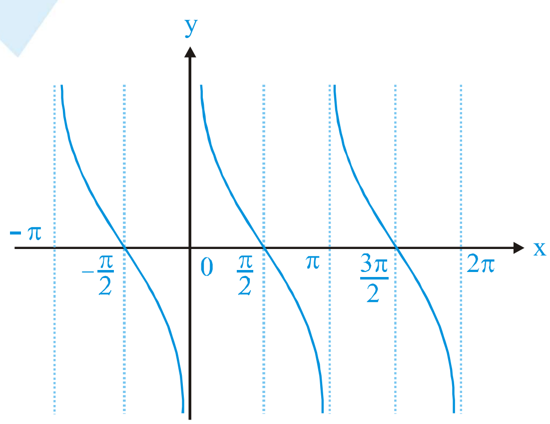

(b) y = cosx



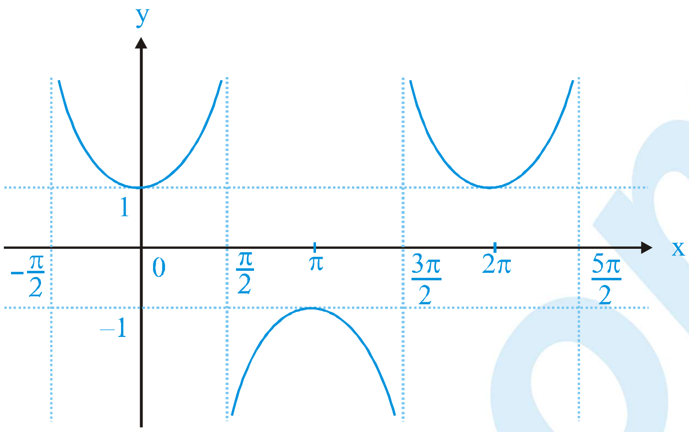

(c) y = tanx



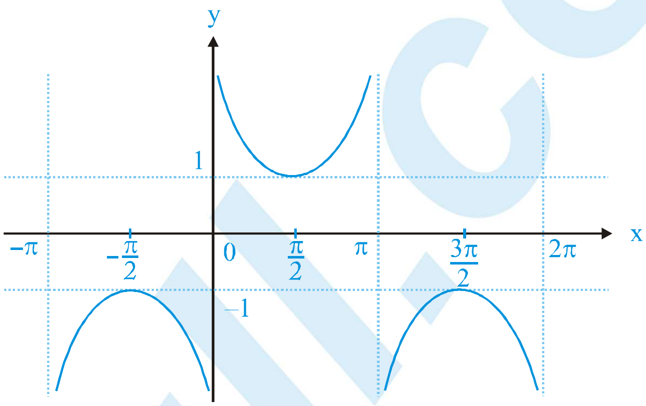

(d) y = cotx



(e) $y = \sec x$




(f) $y = \text{cosec}$




## 12. Important Notes

**(a)** The sum of interior angles of a polygon of n-sides $= (n-2) \times 180^o = (n-2)\pi$.

**(b)** Each interior angles of a polygon of n sides

$$= \frac{(n-2)}{n} \times 180^o = \frac{(n-2)}{n}\pi.$$

**(c)** Sum of exterior angles of a polygon of any number of sides $= 360o = 2\pi$.