Functions

DEFINITION :

Function is a rule (or correspondence), from a non empty set A to a non empty set B, that associates each member of A (say a) to a unique member of B (say b). Symbolically, we write $f: A \to B$. We read it as "f is a function from A to B".

Here b, is called the image of a and a is called the pre- image of b under f.

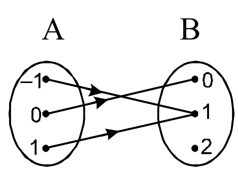
For example, let $A \equiv \{-1, 0, 1\}$ and $B \equiv \{0, 1, 2\}$.

Then $A \times B \equiv \{(-1, 0), (-1, 1), (-1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$

Now, " $f : A \to B$ defined by $f(x) = x^2$ " is the function such that

$f \equiv \{(-1, 1), (0, 0), (1, 1)\}$

f can also be shown diagramatically by following mapping.

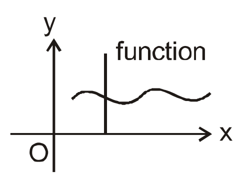


Note :

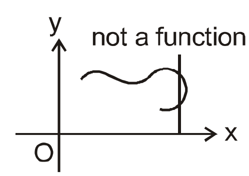
Every function say $y = f(x) : A \to B$. Here x is independent variable which takes its values from A while 'y' takes its value from B. A relation will be a function if and only if

(i) x must be able to take each and every value of A

(ii) one value of x must be related to one and only one value of y in set B.



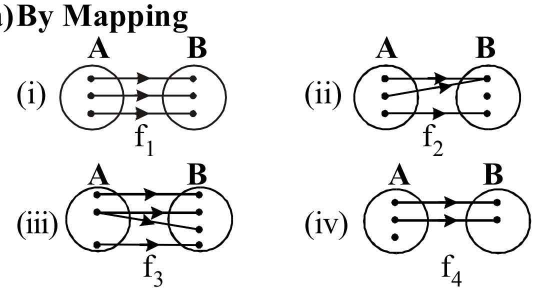
(a)



(b)

Testing for a function : A relation $f : A \to B$ is a function or not, it can be checked by following methods.

(a) By Mapping



In the above given mappings rule f_1 and f_2 shows a function because each element of A is associated with a unique element of B. Whereas f_3 and f_4 are not function because in f_3, one element of A is associated with two elements of B, and in f_4 , one element of A is not associated with any element of B, which do not follow the definition of function.

(b) In the form of ordered pairs :

A function $f : A \to B$ can be expressed as a set of ordered pairs in which first element of every ordered pair is a member of A and second element is the member of B. So f is a set of ordered pairs (a, b) such that :

(i) a is an element of A

(ii) b is an element of B

(iii) Two ordered pairs should not have the same first element.

Solved Examples

Ex.19 If A and B are two sets such that A = {1, 2, 3} and B = {5,7,9} and if a function is defined from $f : A \to B$, $f(x) = 2x + 3$ then find a function in the form of ordered pairs.

Sol. $f : A \to B$, $f(x) = 2x + 3$

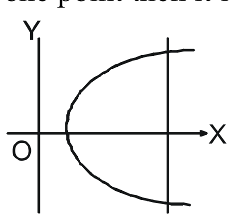
$f(1) = 2.1 + 3 = 5$,

$f(2) = 2.2 + 3 = 7$

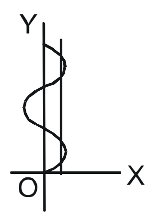
$f(3) = 2.3 + 3 = 9$

$\therefore$ f : {(1,5) ; (2,7) ,(3,9) }

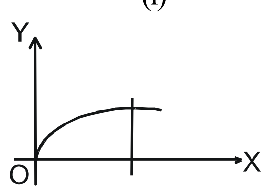
(c) By Vertical Line Test : If we are given a graph of the relation then we can check whether the given relation is function or not . If it is possible to draw a vertical line which cuts the given curve at more than one point then given relation is not a function and when this vertical line means line parallel to Y - axis cuts the curve at only one point then it is a function.



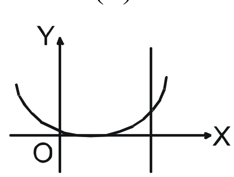
(i)



(ii)



(iii)



(iv)

fig. (iii) and (iv) represents a function.

Solved Examples

Ex.20 Which of the following correspondences can be called a function ?

(A) $f(x) = x^3$; $\{-1, 0, 1\} \to \{0, 1, 2, 3\}$

(B) $f(x) = \sqrt{x}$; $\{0, 1, 4\} \to \{-2, -1, 0, 1, 2\}$

Sol. In case of (A) the given relation is not a function, as $f(-1) \notin 2^{nd}$ set. Hence definition of function is not satisfied, in case of (B) also, the given relation is not a function, as $f(1) = \pm 1$ and $f(4) = \pm 2$ i.e. element 1 as well as 4 in 1^{st} set is related with two elements of 2^{nd} set.Hence definition of function is not satisfied.

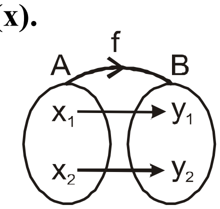
Domain, Co-domain and Range of a Function:

If a function f is defined from a set of A to set B then for $f: A \to B$ set A is called the **domain** of function f and set B is called the **co-domain** of function f. The set of the f- images of the elements of A is called the **range** of function f.

In other words, we can say

Domain = All possible values of x for which f(x) exists.

Range = For all values of x, all possible values of f(x).



Domain = $\{x_1, x_2\}$ = A

Co-domain = $\{y_1, y_2\}$ = B

Range = $\{y_1, y_2\}$ = Co-domain = B

A function whose domain and range are both subsets of real numbers is called **a real function**.

METHODS FOR FINDING DOMAIN

(a) For domain of $y = \dfrac{1}{f(x)}$, the condition is $f(x) \neq 0$

(b) For domain of $y = \sqrt{f(x)}$, the condition is $f(x) \geq 0$

(c) For domain of $y = \dfrac{1}{\sqrt{f(x)}}$, the condition is $f(x) > 0$

(d) For domain of $y = \log(f(x))$, the condition is $f(x) > 0$

(e) For domain of $y = \sin^{-1}(f(x))$ or $\cos^{-1}(f(x))$, the condition is $-1 \leq f(x) \leq 1$.

(f) For domain of $\phi(x) = \{f(x)\}^{g(x)}$, conventionally, the conditions are $f(x) > 0$ and $g(x)$ must be real.

(g) For domain of $\phi(x) = {}^{f(x)}C_{g(x)}$ or $\phi(x) = {}^{f(x)}P_{g(x)}$ conventional conditions of domain are $f(x) \ge g(x)$ and $f(x) \in N$ and $g(x) \in W$

Solved Examples

Ex.21 Find the domain of following functions

$f(x) = \sqrt{x^2 - 5}$

Sol. $f(x) = \sqrt{x^2 - 5}$ is real if f $x^2 - 5 \ge 0$

$\Rightarrow |x| \ge \sqrt{5} \quad \Rightarrow \quad x \le -\sqrt{5}$ or $x \ge \sqrt{5}$

$\therefore$ the domain of f is $(-\infty, -\sqrt{5}\,] \cup [\,\sqrt{5}, \infty)$

Ex.22 If $f : N \to W$, $f(x) = 2x + 3$ is a function, then find domain, co-domain and range of function.

Sol. Domain of f = N

Co-domain of f = W and

Range of f = {5,7,9,11,...}

Ex.23 Find the domain of the function

$f(x) = \sqrt{1-x}$.

Sol. $1 - x \ge 0 \Rightarrow 1 \ge x$

$\therefore x \in (-\infty, 1]$

Ex.24 Find the domain of the function

$$f(x) = \sqrt{\log\left(\frac{5x - x^2}{6}\right)}.$$

Sol. For a given function to be defined

$$\log\left(\frac{5x - x^2}{6}\right) \ge 0 \Rightarrow \frac{5x - x^2}{6} \ge 1$$

$\therefore 5x - x^2 \ge 6 \Rightarrow x^2 - 5x + 6 \le 0$

$(x - 3)(x - 2) \le 0$

$\therefore x \in [2,3]$

Methods of determining range :

(i) Representing x in terms of y

If $y = f(x)$, try to express as $x = g(y)$, then domain of $g(y)$ represents possible values of y, which is range of $f(x)$.

Solved Examples

Ex.25 Find the range of $f(x) = \dfrac{x^2 + x + 1}{x^2 + x - 1}$

Sol. $f(x) = \dfrac{x^2 + x + 1}{x^2 + x - 1}$ $\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor$\}$

$$y = \frac{x^2 + x + 1}{x^2 + x - 1}$$

$\Rightarrow yx^2 + yx - y = x^2 + x + 1$

$\Rightarrow (y - 1)x^2 + (y - 1)x - y - 1 = 0$

If $y = 1$, then the above equation reduces to $-2 = 0$. Which is not true.

Further if $y \ne 1$, then $(y - 1)x^2 + (y - 1)x - y - 1 = 0$ is a quadratic and has real roots if

$(y - 1)^2 - 4(y - 1)(-y - 1) \ge 0$

i.e. if $y \le -3/5$ or $y \ge 1$ but $y \ne 1$

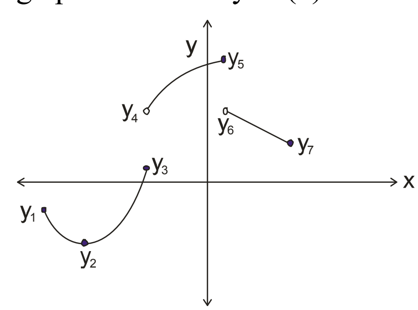
Thus the range is $(-\infty, -3/5] \cup (1, \infty)$

(ii) Graphical Method :

The set of y– coordinates of the graph of a function is the range.

Solved Examples

Ex. 26 Let graph of function $y = f(x)$ is



Then range of above sectionally continuous function is $[y_2, y_3] \cup (y_4, y_5] \cup (y_7, y_6]$

(iii) Using monotonocity :

Many of the functions are monotonic increasing or monotonic decreasing. In case of monotonic continuous functions the minimum and maximum values lie at end points of domain. Some of the common function which are increasing or decreasing in the interval where they are **continuous** is as under.

Monotonic increasing	Monotonic decreasing
$\log_a x,\ a > 1$	$\log_a x,\ 0 < a < 1$
e^x	e^{-x}
$\sin^{-1} x$	$\cos^{-1} x$
$\tan^{-1} x$	$\cot^{-1} x$
$\sec^{-1} x$	$\operatorname{cosec}^{-1} x$

For monotonic increasing functions in [a, b]

(i) $f'(x) \geq 0$ (ii) range is $[f(a), f(b)]$

for monotonic decreasing functions in [a, b]

(i) $f'(x) \leq 0$ (ii) range is $[f(b), f(a)]$

Algebraic Operations on Functions :

Let f and g be two given functions and their domain are D_f and D_g respectively, then the sum, difference, product and quotient functions are defined as :

(a) $(f + g)(x) = f(x) + g(x),\ \forall\ x \in D_f \cap D_g$

(b) $(f - g)(x) = f(x) - g(x),\ \forall\ x \in D_f \cap D_g$

(c) $(f\,.\,g)(x) = f(x)\,.\,g(x),\ \forall\ x \in D_f \cap D_g$

(d) $(f/g)(x) = \dfrac{f(x)}{g(x)}$; $g(x) \neq 0,\ \forall\ x \in D_f \cap D_g$

Solved Examples

Ex.27 Find the domain of following functions :

(i) $f(x) = \sqrt{\sin x} - \sqrt{16 - x^2}$

(ii) $f(x) = \dfrac{3}{\sqrt{4 - x^2}} \log(x^3 - x)$

Sol. (i) $\sqrt{\sin x}$ is real iff $\sin x \geq 0 \Leftrightarrow x \in [2n\pi, 2n\pi + \pi]$, $n \in I$.

$\sqrt{16 - x^2}$ is real iff $16 - x^2 \geq 0 \Leftrightarrow -4 \leq x \leq 4$.

Thus the domain of the given function is $\{x : x \in [2n\pi, 2n\pi + \pi], n \in I\} \cap [-4, 4] = [-4, -\pi] \cup [0, \pi]$.

(ii) Domain of $\sqrt{4 - x^2}$ is $[-2, 2]$ but $\sqrt{4 - x^2} = 0$ for $x = \pm\ 2$ $\Rightarrow$ $x \in (-2, 2)$

$\log(x^3 - x)$ is defined for $x^3 - x > 0$ i.e. $x(x - 1)(x + 1) > 0$.

$\therefore$ domain of $\log(x^3 - x)$ is $(-1, 0) \cup (1, \infty)$.

Hence the domain of the given function is $\{(-1, 0) \cup (1, \infty)\} \cap (-2, 2) \equiv (-1, 0) \cup (1, 2)$.

Vxarious Types of Functions :

Type I Explicit & Implicit Function

(i) Explicit Function :

A function $y = f(x)$ is said to be an explicit function of x if the dependent variable y can be expressed in terms of independent variable x only.

Example -(i) $y = x - \cos x$ (ii) $y = x + \log x - 2x^3$

(ii) Implicit function :

A function $y = f(x)$ is said to an implicit function of x if y cannot be written in terms of x only.

Example - (i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

(ii) $xy = \text{Sin}(x + y)$

Type II Algebraic Function :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form, $P_0(x)\,y^n + P_1(x)\,y^{n-1} + \ldots\ldots + P_{n-1}(x)\,y + P_n(x) = 0$, where n is a positive integer and $P_0(x), P_1(x)\ldots\ldots$ are polynomials in x. e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note :

All polynomial functions are algebraic but not the converse.

(a) Polynomial Function :

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$, where n is a **non negative integer** and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

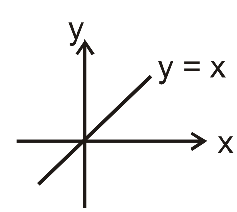
» $f(x) = C$, is a polynomial of zero power or a constant function.

» $f(x) = a x + b$, is a polynomial of power one or a linear function.

» $f(x) = ax^2 + bx + c$, is a polynomial of two power or a quadratic function and so on.

(i) Identity function :

The function $f : A \to A$ defined by, $f(x) = x, \forall\ x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.



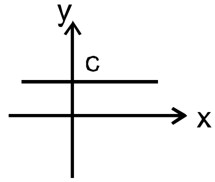
Domain = R Range = R

(ii) Constant function :

A function $f: A \to B$ is said to be a constant function, if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c$, $\forall\ x \in A$, $c \in B$ is a constant function.

Domain = R

Range = C



(b) Rational Function :

A rational function is a function of the form, $y = f(x) = \frac{g(x)}{h(x)}$, where g (x) and h (x), $h(x) \neq 0$ are polynomials.

(c) Irrational Function:

A function which is not rational is called Irrational Function.

Solved Examples

Ex.28 $f(x) = x^2 + x + \sqrt{x} + 5$, $g(x) = \frac{x^2 - \sqrt{x}}{1 + x^{1/3}}$ etc.

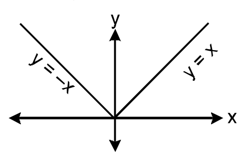
(d) Piecewise Functions

(i) Absolute Value Function / Modulus Function :

The symbol of modulus function is $f(x) = |x|$ and is defined as: $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

Domain = R

Range = $[0, \infty)$

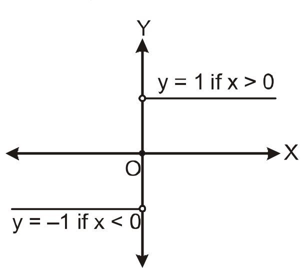


y = |x|

(ii) Signum Function : (Also known as sgn(x))

A function $f(x) = \text{sgn}(x)$ is defined as follows :

$$f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



y = sgn x

It is also written as $\text{sgn}\, x = \begin{cases} \frac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$

Domain = R

Range = $\{-1, 0, 1\}$

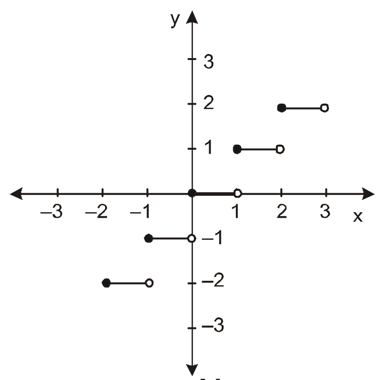
Note : $\text{sgn}\, f(x) = \begin{cases} \frac{|f(x)|}{f(x)}; & f(x) \neq 0 \\ 0; & f(x) = 0 \end{cases}$

(iii) Greatest Integer Function or Step Function or Box Function :

The function $y = f(x) = [x]$ is called the greatest integer function, where [x] equals to the greatest integer less than or equal to x. For example :

for $-1 \leq x < 0$; $[x] = -1$; for $0 \leq x < 1$; $[x] = 0$

for $1 \leq x < 2$; $[x] = 1$; for $2 \leq x < 3$; $[x] = 2$ and so on.



y = [x]

Domain = R

Range = I

Note : Important Identities :

(i) $[x] \leq x$ (This is always true)

(ii) $[x] + 1 > x$

Solved Examples

Ex.29 Find the range of $f(x) = x - [x]$

Sol. $\because$ we know $x - [x] = \{x\}$:

{where {x} is fractional part function} and $0 \leq \{x\} < 1$

$\therefore$ Range of f(x) is [0, 1)

Ex.30 Find the range of f(x) = 3 + x – [x+2]

Sol. f(x) = 3 + x – [x + 2]

= 1 + 2 + x – [x + 2] {where {.} is fractional part function}

= 1 + {2 + x}

$\because$ 0 ≤ {2 + x} < 1

$\therefore$ 0 +1 ≤ {2 + x} + 1 < 1 + 1

$\therefore$ 1 ≤ f(x) < 2

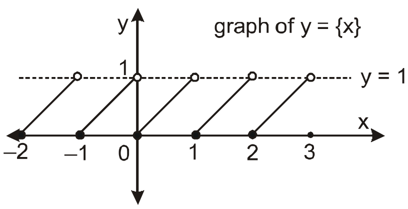
$\therefore$ Range of f(x) is [1, 2)

(iv) Fractional Part Function:

It is defined as, y = {x} = x – [x], where [.] denotes greatest integer function.

e.g. the fractional part of the number 2.1 is 2.1 – 2 = 0.1 and {–3.7} = 0.3.

The period of this function is 1 and graph of this function is as shown.



Domain = R

Range = [0, 1)

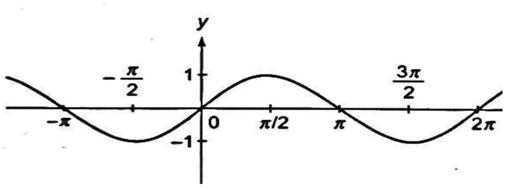
Type III Transcendental Function.

A function that is not algebraic is called Transcendental Function.

(a) Trigonometric Functions :

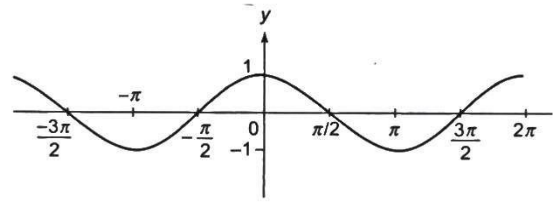
(i) Y = f(x) = sin x

Domain $\rightarrow$R Range $\rightarrow$[-1,1] Period $A \rightarrow 2\pi$



(ii) Y = f(x) = cosx

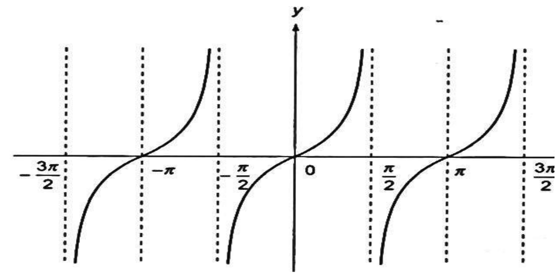
Domain $\rightarrow$R Range $\rightarrow$[-1,1] Period $A \rightarrow 2\pi$



(iii) Y = f (x) = tanx

Domain $\rightarrow R \sim (2n+1)\dfrac{\pi}{2}, n \in I$

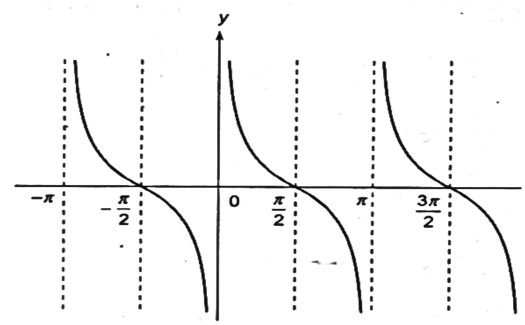
Range $\rightarrow (-\infty, \infty)$ Period $\rightarrow \pi$



(iv) Y = f (x) = cotx

Domain $\rightarrow R \sim n\pi, n \in I$

Range $\rightarrow (-\infty, \infty)$

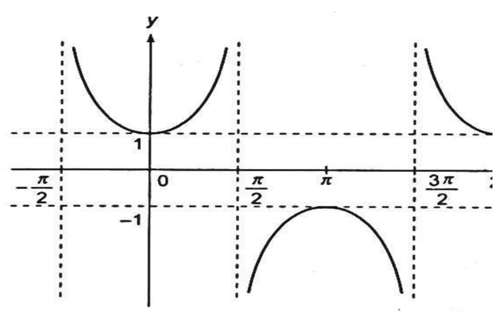


(v) Y = f (x) = secx

Domain $\rightarrow R - (2n+1)\dfrac{\pi}{2}, n \in I$

Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Period $\rightarrow 2\pi$, $\sec^2 x$, |secx| ∈ [1, ∞)

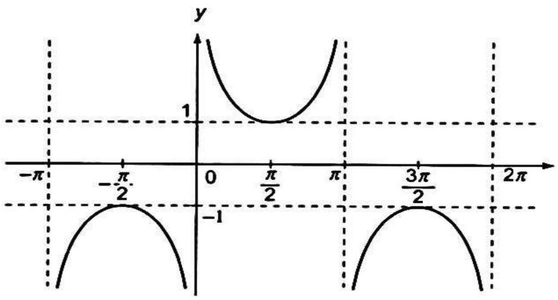


(vi) $Y = f(x) = \text{cosec}x$

Domain $\rightarrow R - n\pi,\ n \in I$

Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

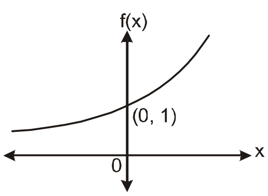
Period $\rightarrow 2\pi$, cosec^2x, $|\text{cosec}x| \in [1, \infty)$



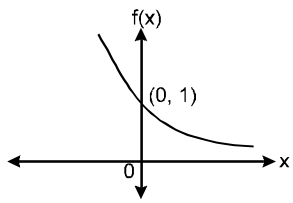
(b) Exponential Function :

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0,\ a \neq 1,\ x \in R)$ is called an exponential function. Graph of exponential function can be as follows :

Case - I
For a > 1



Case - II
For 0 < a < 1

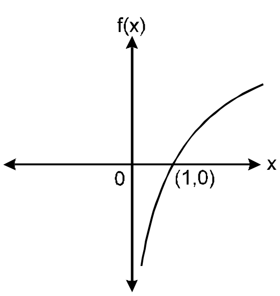


Domain = R

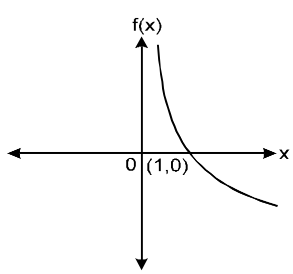
Range = $(0, \infty)$

(c) Logarithmic Function : $f(x) = \log_a x$ is called logarithmic function, where $a > 0$ and $a \neq 1$ and $x > 0$. Its graph can be as follows

Case- I
For a > 1



Case- II
For 0 < a < 1



Domain = $[0, \infty)$

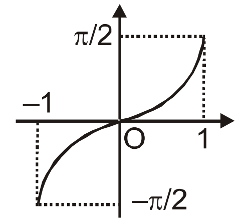
Range = R

(d) Inverse Trigonometric Function :

$y = \sin^{-1} x$

Domain = $[-1, 1]$

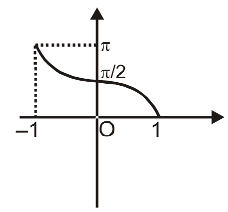
Range $= \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



$y = \cos^{-1}x$

Domain = $[-1, 1]$

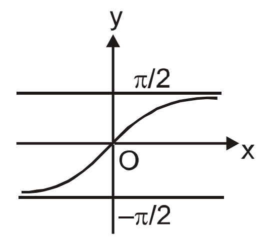
Range = $[0, \pi]$



$y = \tan^{-1}x$

Domain = R

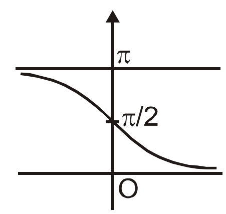
Range $= \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$



$y = \cot^{-1} x$

Domain = R

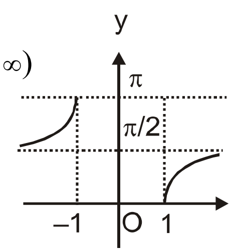
Range = $(0, \pi)$



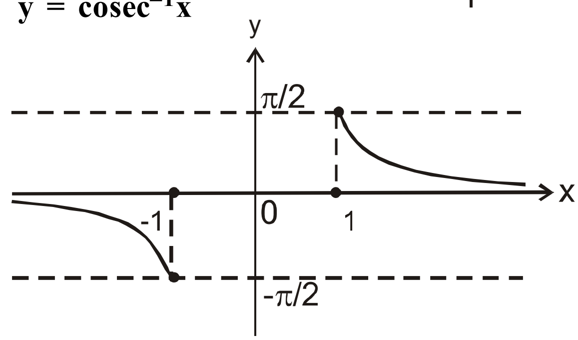
$y = \sec^{-1}x$

Domain = $(-\infty, -1] \cup [1, \infty)$

Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$



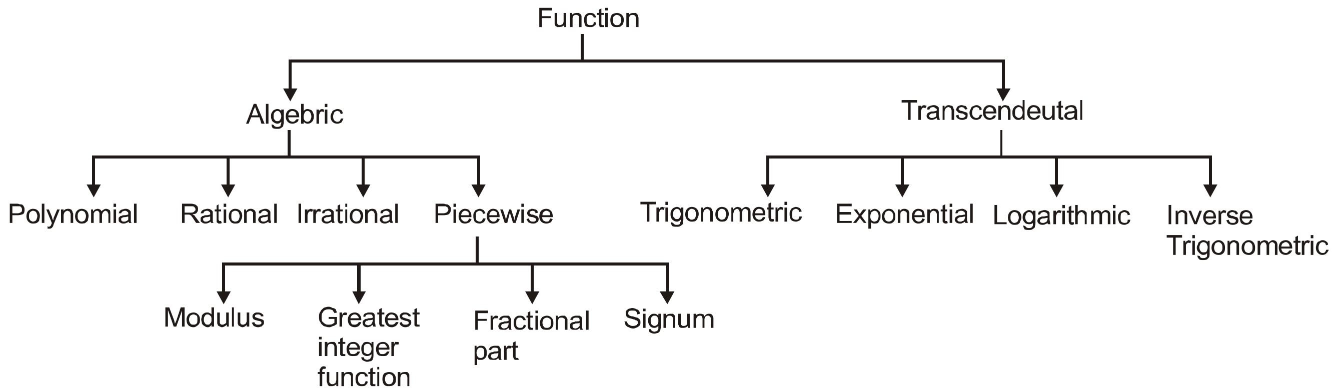
$y = \text{cosec}^{-1}x$



Domain = $(-\infty, -1] \cup [1, \infty)$

Range $= \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

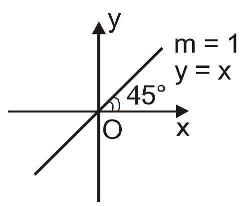
Classification of function :



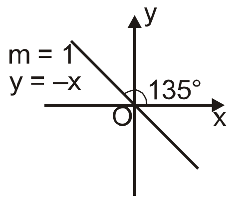
Graphs of some more basic functions

(A) $y = f(x) = mx$, where m is a constant $(m \neq 0)$

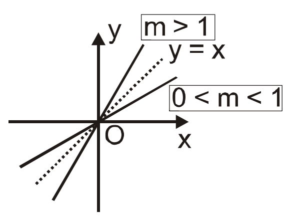
(i)



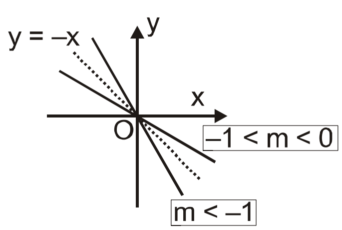
(ii)



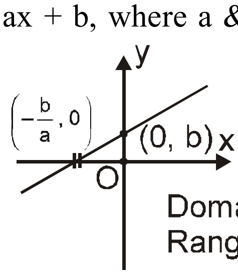
(iii)



(iv)

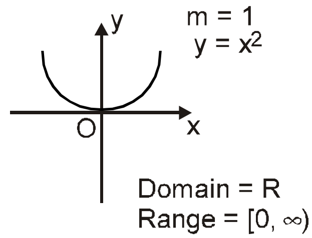


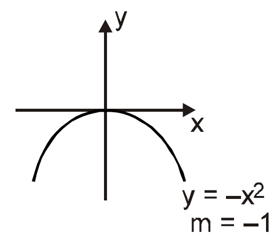
(v) $y = ax + b$, where a & b are constant



Domain = R
Range = R

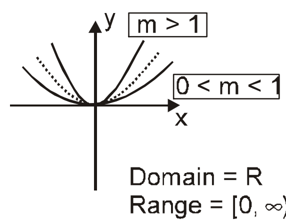
(B) $y = mx^2$, $(m \neq 0)$, $x \in R$

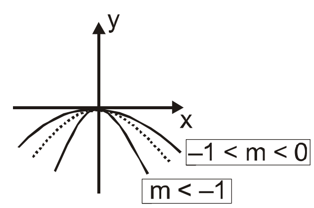




Domain = R

Range = $(-\infty, 0]$



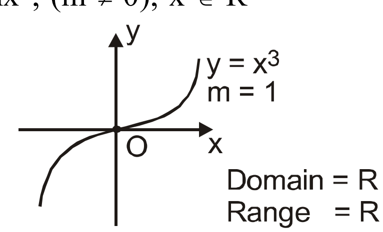


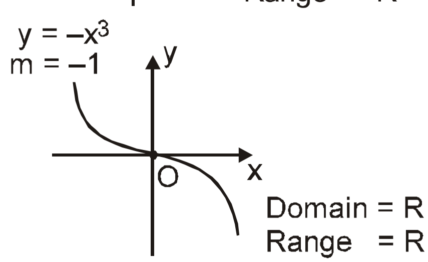
Domain = R

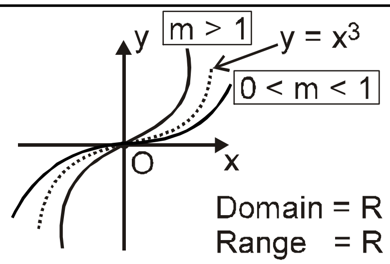
Range $= [0, \infty)$

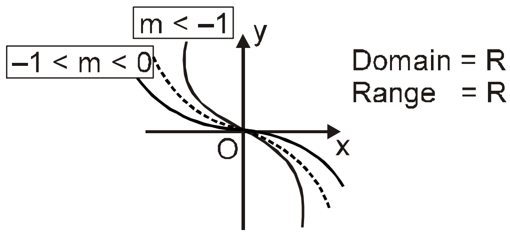
Graph is symmetric about y-axis

(C) $y = mx^3$, $(m \neq 0)$, $x \in R$



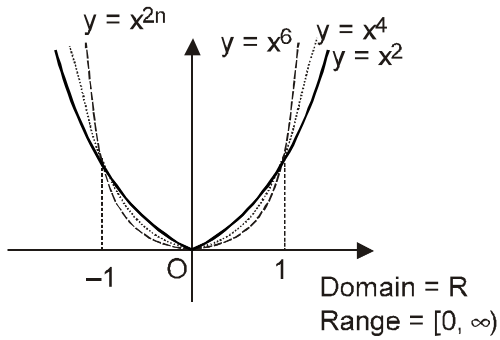






Graph is symmetric about origin.

(D) $y = x^{2n}$, $n \in N$



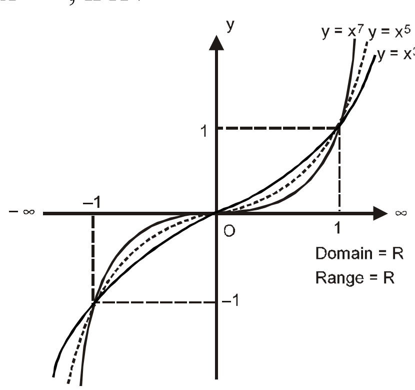
From the graph :

For $|x| \le 1 \Rightarrow x^6 \le x^4 \le x^2$

For $|x| > 1 \Rightarrow x^6 > x^4 > x^2$

Graph is symmetric about y-axis

(E) $y = x^{2n+1}$, $n \in N$



From the graph :

For $x \in (-\infty, -1] \cup [0, 1]$

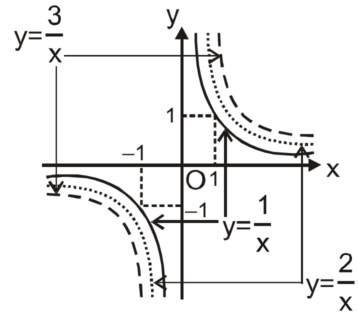
$\Rightarrow x^3 \ge x^5 \ge x^7$

For $x \in (-1, 0) \cup (1, \infty)$

$\Rightarrow x^3 < x^5 < x^7$

Graph is symmetric about origin.

(F) $y = \dfrac{m}{x}$, $x \ne 0$, $m \ne 0$

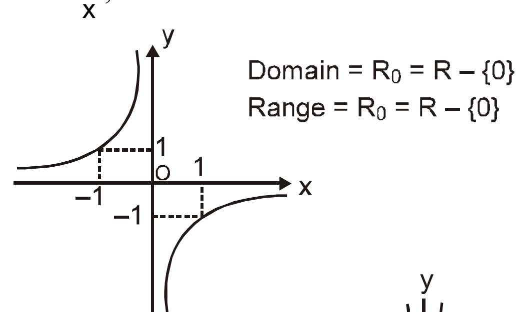


Domain = R_0 = R – {0}

Range = R_0 = R – {0}

This type of graph xy = m is called rectangular hyperbola.

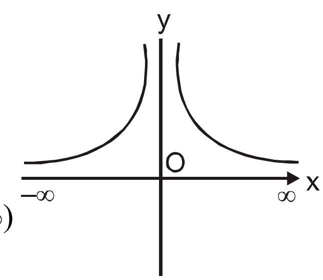
(G) $y = -\dfrac{1}{x}$, $x \ne 0$



(H) $y = \dfrac{1}{x^2}$, $x \ne 0$

Domain = R_0

Range = R^+ = (0, ∞)

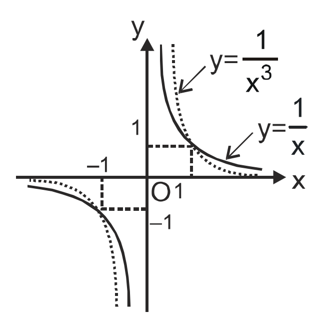


(I) $y = \dfrac{1}{x^{2n-1}}$, $n \in N$

From graph

$0 < x < 1 \Rightarrow \dfrac{1}{x^3} > \dfrac{1}{x}$

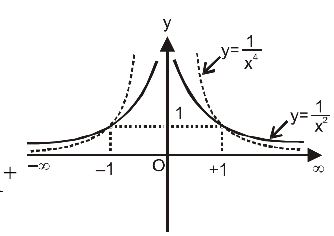
$x > 1 \Rightarrow \dfrac{1}{x^3} < \dfrac{1}{x}$.



(J) $y = \dfrac{1}{x^{2n}}$, $n \in N$

Domain = R_0

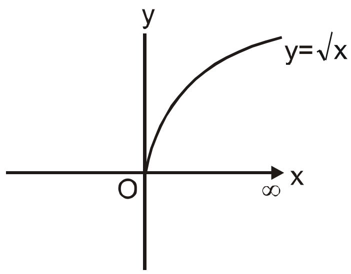
Range = (0, ∞) = R^+



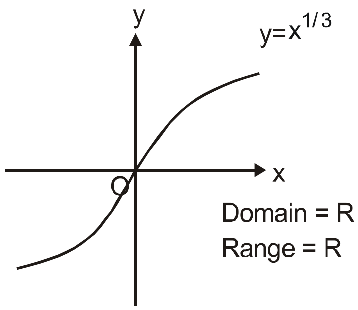
(K) $y = \sqrt{x} = x^{1/2}$

Domain = $[0, \infty)$

Range = $[0, \infty)$



(L) $y = \sqrt[3]{x} = x^{1/3}$



(M) $y = x^{1/2n}$, $n \in N$

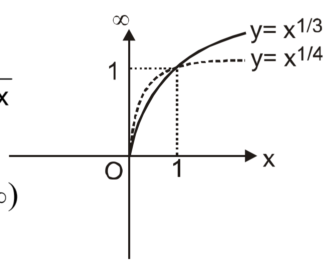
From graph

$0 < x < 1 \Rightarrow \sqrt[2]{x} > \sqrt[4]{x}$

$x > 1 \Rightarrow x^{1/2} < x^{1/4}$

Domain = $R^{+} = [0, \infty)$

Range = $R^{+} = [0, \infty)$



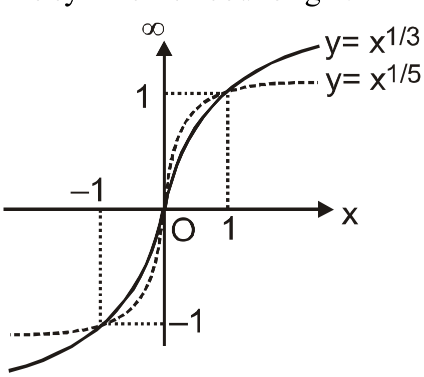
(N) $y = x^{\frac{1}{2n+1}}$, $n \in N$

From graph

$x \in (-\infty, -1) \cup (0, 1)$ $x^{1/3} > x^{1/5}$

$x \in (-1, 0) \cup (1, \infty)$ $x^{1/3} < x^{1/5}$

Graph is symmetric about origin.



Domain = R

Range = R

Odd and Even Functions :

(i) If $f(-x) = f(x)$ for all x in the domain of 'f', then f is said to be an even function.

e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.

(ii) If $f(-x) = -f(x)$ for all x in the domain of 'f', then f is said to be an odd function.

e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

Note :

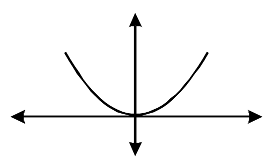
(i) A function may neither be odd nor even. (e.g. $f(x) = e^x$, $\cos^{-1}x$)

(ii) If an odd function is defined at $x = 0$, then $f(0) = 0$

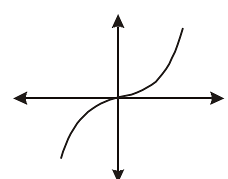
Properties of Even/Odd Function

(a) The graph of every even function is symmetric about the y–axis and that of every odd function is symmetric about the origin.

For example graph of $y = x^2$ is symmetric about y-axis, while graph of $y = x^3$ is symmetric about origin



$y = x^2$



$y = x^3$

(b) All functions (whose domain is symmetrical about origin) can be expressed as the sum of an even and an odd function, as follows

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

(c) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(d) If f and g both are even or both are odd, then the function f.g will be even but if any one of them is odd and the other even then f.g will be odd.

(e) If f(x) is even then f′(x) is odd while derivative of odd function is even. Note that same cannot be said for integral of functions.

Solved Examples

Ex.31 Show that $\log\left(x+\sqrt{x^2+1}\right)$ is an odd function.

Sol. Let $f(x)=\log\left(x+\sqrt{x^2+1}\right)$.

Then $f(-x)=\log\left(-x+\sqrt{(-x)^2+1}\right)$

$$=\log\left(\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\sqrt{x^2+1}+x}\right)$$

$$=\log\frac{1}{\sqrt{x^2+1}+x}=-\log\left(x+\sqrt{x^2+1}\right)$$

$=-f(x)$ or $f(x)+f(-x)=0$

Hence f(x) is an odd function.

Ex.32 Show that a^x+a^{-x} is an even function.

Sol. Let $f(x) = a^x + a^{-x}$

Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$.

Hence f(x) is an even function

Ex.33 If $f(x) = \dfrac{a^x+1}{a^x-1}$ then find whether it is an even or odd function .

Sol. $f(-x) = \dfrac{a^{-x}+1}{a^{-x}-1} = -\dfrac{a^x+1}{a^x-1}$

$\therefore$ $f(-x) = -f(x)$

$\therefore$ f(x) is an odd function.

Ans.

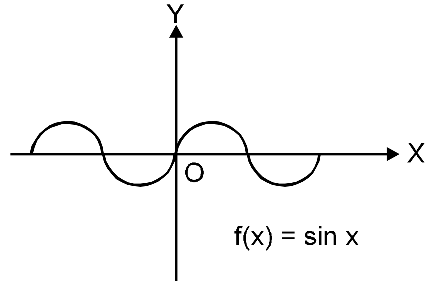
Continuous and Discontinuous Function

(a) Continuous Function :

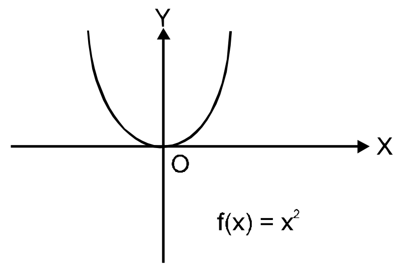
A function is said to be continuous function in an interval I if we are not required to lift the pen or pencil off the paper i.e. there is no gap or break or jump in the graph.

Solved Examples

Ex.34 $f(x) = x^2$, $f(x) = \sin x$, $f(x) = |x|$, $f(x) = \cos x$ all are continuous function.



(continuous function)



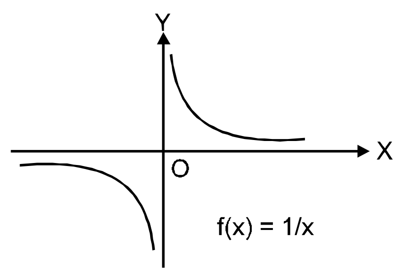
(continuous function)

(b) Discontinuous Function :

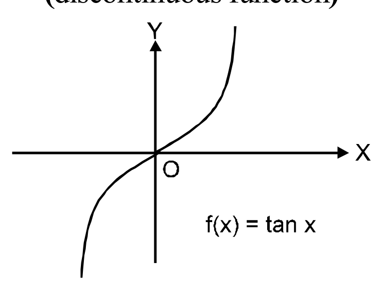
A function is said to be discontinuous if there is a break or gap or jump in the graph of the function at any point.

Solved Examples

Ex.35 $f(x) = 1/x$, $f(x) = \tan x$, $f(x) = [x]$ are discontinuous functions.



(discontinuous function)



(discontinuous function)

Increasing and Decreasing Function :

(a) Increasing Function :

A function f(x) is called increasing function in the domain D if the value of the function does not decrease by increasing the value of x.

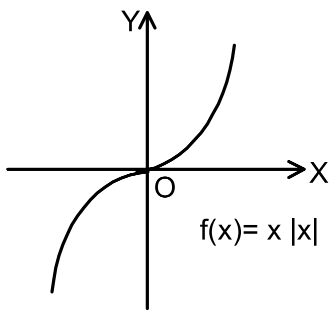
so $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \ \forall x_1, x_2 \in$ domain

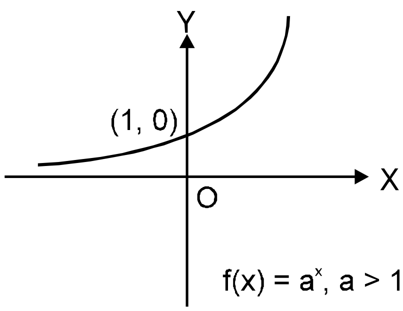
or $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \ \forall x_1, x_2 \in$ domain

Solved Examples

Ex.36 $f(x) = e^x$, $f(x) = a^x$, $f(x) = x^2$, $x \geq 0$, $f(x) = x|x|$ are increasing functions.

The graph of these functions rises from left to right.





A function is called strictly increasing if

if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \ \forall x_1, x_2 \in$ domain

(b) Decreasing Function :

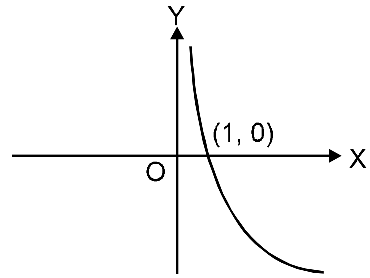
A function f (x) is said to be decreasing function in the domain D if the value of the function does not increase by increasing the value of x (variable).

so if $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \ \forall x_1, x_2 \in D$

Solved Examples

Ex.37 $f(x) = \log_a x$ $(a < 1)$, $f(x) = e^{-x}$ are decreasing functions. The graph of these functions is downward from left to right.



$f(x) = \log_a x$ $(a < 1)$

A function is called strictly decreasing if

if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

or $x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \ \forall x_1, x_2 \in D$

Note :

It is not essential for any function to be increasing or decreasing. There are some functions which are neither increasing nor decreasing i.e. function is increasing in one part of given interval and decreasing in second part.

Solved Examples

Ex.38 $f(x) = \sin x$, $f(x) = |x|$, $f(x) = e^x + e^{-x}$

Periodic Functions :

A function f(x) is called periodic with a period T if there exists a real number T > 0 such that for each x in the domain of f the numbers x – T and x + T are also in the domain of f and f(x) = f(x + T) for all x in the domain of f(x). Graph of a periodic function with period T is repeated after every interval of 'T'.

e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π.

The least positive period is called the principal or fundamental period of f(x) or simply the period of the function.

Note : Inverse of a periodic function does not exist.

Properties of Periodic Functions :

(a) If f(x) has a period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(b) If f(x) has a period T, then f(ax + b) has a period $\frac{T}{|a|}$.

(c) Every constant function defined for all real x, is always periodic, with no fundamental period.

(d) If f(x) has a period T_1 and g(x) also has a period T_2 then period of f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is L.C.M. of T_1 and T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is nonperiodic.

L.C.M. of $\left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{L.C.M.}(a, p, \ell)}{\text{H.C.F.}(b, q, m)}$

e.g. |sinx| has the period π, | cosx | also has the period π

∴ |sinx| + |cosx| also has a period π. But the fundamental period of |sinx| + |cosx| is $\frac{\pi}{2}$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

Solved Examples

Ex.39 Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$

(ii) f(x) = {x} + sin x, where {.} denotes fractional part function

(iii) f(x) = cos x . cos 3x

(iv) $f(x) = \sin\frac{3x}{2} - \cos\frac{x}{3} - \tan\frac{2x}{3}$

Sol. (i) Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π.

Hence period of $\sin\frac{x}{2} + \cos\frac{x}{3}$ is 12 π

{L.C.M. of 4 and 6 is 12}

(ii) Period of sin x = 2π

Period of {x} = 1 but L.C.M. of 2π and 1 is not possible as their ratio is irrational number

∴ it is aperiodic

(iii) f(x) = cos x . cos 3x

period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period = $\frac{2\pi}{n}$, where n ∈ N. Hence cross-checking for n = 1, 2, 3, ...we find π to be fundamental period f(π + x) = (– cos x) (– cos 3x) = f(x)

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{2/3}$

= L.C.M. of $\frac{4\pi}{3}$, 6π , $\frac{3\pi}{2}$ = 12π

Ex.40 Find the period of |cos x|

Sol. |cos x| = | cos (π + x) | = | cos (2π + x) | = ...

∴ period of |cos x| is π

Second Method :

$f(x) = |\cos x| = \sqrt{\cos^2 x} = \sqrt{\frac{1+\cos 2x}{2}}$

∵ period of cos 2 x = 2 π /2 = π

∴ period of f (x) = π

Ex.41 Find the period of function x – [x] where [x] is a greatest integer function.

Sol. ∵ we know that x – [x] = {x} (Fractional part function) and period of fractional part function is 1

∴ period of f(x) = 1

VALUE OF THE FUNCTION

If y = f(x) is any function defined in R, then for any given value of x (say x = a), the value of the function f(x) can be obtained by substituting x = a in it and it is denoted by f(a).

Solved Examples

Ex.42 If $f(x) = x^2 - x + 1$ then find f(0) and f(2).

Sol. $f(0) = 0^2 - 0 + 1 = 1$, $f(2) = 2^2 - 2 + 1 = 3$

Ex.43 If $f(x) = \frac{x^2+1}{x-2}$ then find f(0), f(1) and f(2).

Sol. $f(0) = \frac{0+1}{0-2} = \frac{-1}{2}$, $\quad f(1) = \frac{1+1}{1-2} = -2$

$f(2) = \frac{2^2+1}{2-2} = \frac{5}{0}$ which does not exist

Ex.44 If $f(x) = \begin{cases} 4x+3 & 0 < x < 1 \\ 6-2x & 1 \le x < 4 \end{cases}$ then find f(1/2) and f(2).

Sol. $f(1/2) = 4.\ 1/2 + 3 = 5$, $f(2) = 6 - 2.2 = 2$

Note :

The value of f(0), f(–1), f(4), f(5) all will be undefined, because function is only defined in the interval (0,4).

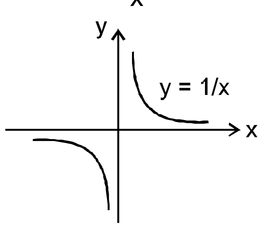
Equal or Identical Functions :

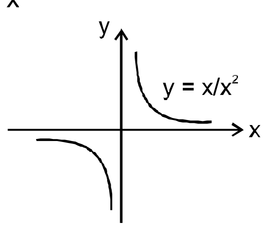
Two functions f and g are said to be identical (or equal) iff :

(i) The domain of $f \equiv$ the domain of g.

(ii) $f(x) = g(x)$, for every x belonging to their common domain.

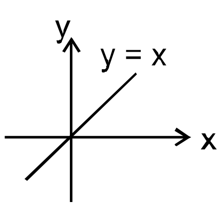
e.g. $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ are identical functions.

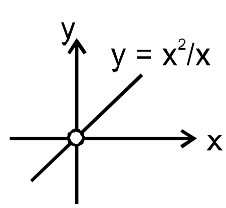




Clearly the graphs of f(x) and g(x) are exactly same

But $f(x) = x$ and $g(x) = \frac{x^2}{x}$ are not identical functions.





Clearly the graphs of f(x) and g(x) are different at $x = 0$.

Solved Examples

Ex.45 Examine whether following pair of functions are identical or not ?

(i) $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$

(ii) $f(x) = \sin^2 x + \cos^2 x$ and $g(x) = \sec^2 x - \tan^2 x$

Sol. (i) No, as domain of f(x) is $R - \{1\}$

while domain of g(x) is R

(ii) No, as domain are not same. Domain of f(x) is R

while that of g(x) is $R - \left\{(2n+1)\frac{\pi}{2}; n \in I\right\}$

Types of Functions (Mapings) :

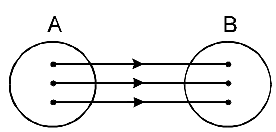
Functions can be classified as "One – One Function (Injective Mapping)" and "Many – One Function":

One - One Function :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \ne x_2 \Leftrightarrow f(x_1) \ne f(x_2)$.

Diagrammatically an injective mapping can be shown as



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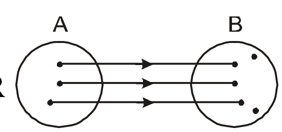
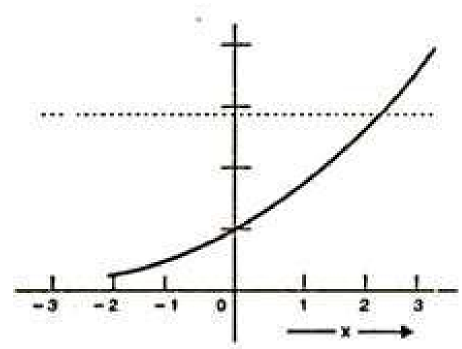


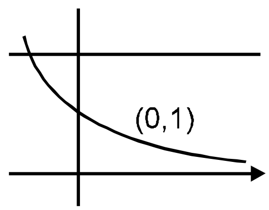


Note :

(a) If $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, equate $f(x_1)$ and $f(x_2)$ and if it implies that $x_1 = x_2$, then and only then function is ONE-ONE otherwise MANY-ONE.

(b) If function is given in the form of ordered pairs and if no two ordered pairs have same second element then function is one-one otherwise MANY-ONE.

(c) If the graph of the function y = f(x) is given, and each line parallel to x-axis cuts the given curve at maximum one point then function is one-one other-wise MANY-ONE.



$f(x) = a^x (0 < a < 1)$

(d) If either $f'(x) \geq 0$, $\forall$ x $\in$ domain or $f'(x) \leq 0$ $\forall$ x $\in$ domain, where equality can hold at discrete point(s) only i.e. strictly monotonic, then function is ONE-ONE, otherwise MANY-ONE.

Examples of One-One Function –

(i) $f : R \to R$, $f(x) = x$,

(ii) $f : R \to R$, $f(x) = ax + b$

(iii) $f : R \to R$, $f(x) = ax^n + b$, n is odd positive integer

(iv) $f : R \to R$, $f(x) = x\,|\,x\,|$

(v) $f : R \to R$, $f(x) = e^x$,

(vi) $f : R \to R$, $f(x) = a^x$ (a > 0)

(vii) $f : R \to R$, $f(x) = \sinh x$,

(viii) $f : R \to R$, $f(x) = \tanh(x)$

(ix) $f : R_0 \to R$, $f(x) = 1/x$,

(x) $f : R^+ \to R$, $f(x) = \log x$,

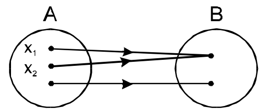
(xi) $f : R_0 \to R$, $f(x) = \log_a x$ (a>0)

Many - One function :

A function $f : A \to B$ is said to be a many one function if there exist at least two or more elements of A having the same f image in B.

Thus $f : A \to B$ is many one iff there exist atleast two elements $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



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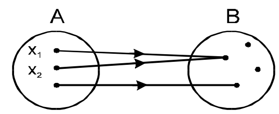
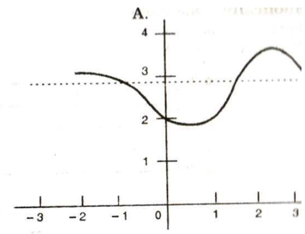




Note :

(1) If a function is one–one, it cannot be many one and vice versa.

(2) All Even Functions & Periodic Functions are many one functions

Example of many-one function :

(i) $f : R \to R$, $f(x) = C$, where C is a constant

(ii) $f : R \to R$, $f(x) = x^2$

(iii) $f : R \to R$, $f(x) = ax^2 + b$,

(iv) $f : R \to R$, $f(x) = |x|$,

(v) $f : R \to R$, $f(x) = x + |x|$

(vi) $f : R \to R$, $f(x) = x - |x|$

(vii) $f : R \to R$, $f(x) = \cosh x$

(viii) $f : R \to R$, $f(x) = [x]$,

(ix) $f : R \to R$, $f(x) = x - [x]$

Where [x] is greatest integer function.

Solved Examples

Ex.46 If A = {2,3,4} and B = {3,4,5,6,7} and a function from A to B is defined by the rule f(x) = 2x –1 then find whether a given function is one-one or not?

Sol. f (2) = 2.2 – 1 = 3, f (3) = 2.3 – 1 = 5

f (4) = 2.4 – 1 = 7

So different elements of A are having different f-images in B, so function is one-one.

Ex.47 If $A = \{-2, 0, 2\}$ and $B = \{1, 10, 17\}$ and a function f from A to B is defined by the rule $f(x) = x^4 + 1$ then find whether the function is one-one or not.

Sol. $f(-2) = (-2)^4 + 1 = 17$

$f(0) = (0)^4 + 1 = 1$

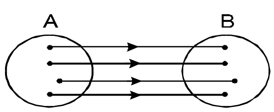
$f(2) = (2)^4 + 1 = 17$

Here $-2 \neq 2$ but $f(-2) = f(2)$ so function is many-one. **Ans.**

Onto function :

If the function $f : A \to B$ is such that each element in B (co–domain) must have atleast one pre–image in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall\, b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Diagrammatically surjective mapping can be shown as



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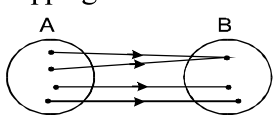


Working procedure for checking the surjective (onto) of a function :

(i) Take an arbitary element y in the co-domain such that $f(x) = y$

(ii) Solve $f(x) = y$, for x and obtain x in the term of y.

(iii) Get the equation of the form $x = g(y)$

(iv) If $x = g(y)$ belong to domain of f, for all value of y then f is onto.

Examples of onto function :

(i) $f : R \to R$, $f(x) = x$,

(ii) $f : R \to R$, $f(x) = ax + b$, $a \neq 0$, $b \in R$

(iii) $f : R \to R$, $f(x) = x^3$

(iv) $f : R \to R$, $f(x) = x|x|$

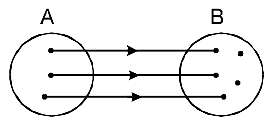
(v) $f : R \to R^+$, $f(x) = e^x$

(vi) $f : R^+ \to R$, $f(x) = \log x$.

Into function :

If $f : A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function can be shown as



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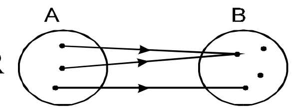


Note :

(i) If range $\equiv$ co–domain, then f(x) is onto, otherwise into

(ii) If a function is onto, it cannot be into and vice versa.

Examples of into function :

(i) $f : R \to R$, $f(x) = x^2$

(ii) $f : R \to R$, $f(x) = |x|$

(iii) $f : R \to R$, $f(x) = c$ (c is constant)

(iv) $f : R \to R$, $f(x) = \sin x$

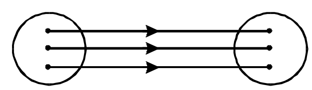
(v) $f : R \to R$, $f(x) = \cos x$

(vi) $f : R \to R$, $f(x) = e^x$

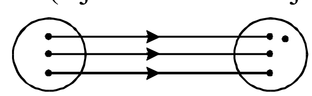
(vii) $f : R \to R$, $f(x) = a^x$, $a > 0$

A function can be one of these four types:

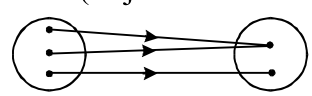
(a) one–one onto (injective and surjective)



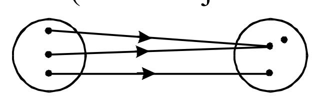
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note :

(i) If f is both injective and surjective, then it is called a **bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains 'n' distinct elements, then the number of different functions defined from $A \to A$ is n^n and out of which $n!$ are one one.

Solved Examples

Ex.48 (i) Find whether f(x) = x + cos x is one-one.

(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$ for $f : R \to R$ is ONTO or INTO

(iii) If $f : R \to R$ be a function defined by $f(x) = 4x^3 - 7$, show that the function f is bijective

Sol. (i) The domain of f(x) is R. $f'(x) = 1 - \sin x$.

$\therefore$ $f'(x) \geq 0 \ \forall \ x \in$ complete domain and equality holds at discrete points only

$\therefore$ f(x) is strictly increasing on R. Hence f(x) is one-one.

(ii) As range $\equiv$ codomain, therefore given function is ONTO

(iii) Given that $f(x) = 4x^3 - 7$, $x \in R$

f is one-one - Let $x_1, x_2 \in R$ (domain)

such that $f(x_1) = f(x_2)$

$\Rightarrow 4x_1^3, -7 = 4x_2^3 - 7$

$\Rightarrow 4x_1^3 = 4x_2^3$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$ $\therefore$ f is one one

f is onto $\Rightarrow$ Let $y \in R$ (co-domain)

such that f(x) = y

$\Rightarrow 4x^3 - 7 = y$

$\Rightarrow x = \left(\dfrac{y+7}{4}\right)^{1/3} \in R$ (domain)

$\Rightarrow \therefore$ f is onto

$\therefore$ Hence f is a bijective function

Composite Function :

Let f: $X \to Y_1$ and g: $Y_2 \to Z$ be two functions and D is the set of values of x such that if $x \in X$, then $f(x) \in Y_2$. If $D \neq \phi$, then the function h defined on D by $h(x) = g\{f(x)\}$ is called composite function of g and f and is denoted by gof. It is also called function of a function.

Note : Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(X) \subseteq Y_2$.

Pictorially gof(x) can be viewed as under

$$\longrightarrow \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$$

Note that gof(x) exists only for those x when range of f(x) is a subset of domain of g(x).

Properties of Composite Functions :

(a) In general gof $\neq$ fog (i.e. not commutative)

(b) The composition of functions are associative i.e. if three functions f, g, h are such that fo (goh) and (fog) oh are defined, then fo (goh) = (fog) oh.

(c) If f and g both are one-one, then gof and fog would also be one-one (if they exist).

(d) If f and g both are onto, then gof or fog may or may not be onto.

(e) The composite of two bijections is a bijection iff f and g are two bijections such that gof is defined, then gof is also a bijection only **when co-domain of f is equal to the domain of g**.

(f) Let $f : A \to B$ and $g : B \to C$ be two function. Then

(i) gof : $A \to C$ is onto and g:$B \to C$ is onto

(ii) gof : $A \to C$ is one-one = f : $A \to B$ is one-one

(iii) gof : $A \to C$ is onto and g: $B \to C$ is one-one $\to$ f : $A \to B$ is onto.

(iv) gof : $A \to C$ is one-one and f: $A \to B$ is onto $\to$ g: $B \to C$ is one-one.

Solved Examples

Ex.49 Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Sol. (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$.

For gof(x)

Since range of f is a subset of domain of g,

$\therefore$ domain of gof is $[-3, \infty)$
{equal to the domain of f}

$gof(x) = g\{f(x)\} = g(\sqrt{x+3}) = 1 + (x+3) = x + 4$. Range of gof is $[1, \infty)$.

For fog(x)

since range of g is a subset of domain of f,

$\therefore$ domain of fog is R
{equal to the domain of g}

$fog(x) = f\{g(x)\} = f(1 + x^2) = \sqrt{x^2+4}$
Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[-1, \infty)$.

For gof(x)

Since range of f is a subset of the domain of g,

$\therefore$ domain of gof is $[0, \infty)$ and $g\{f(x)\} = g(\sqrt{x}) = x - 1$. Range of gof is $[-1, \infty)$

For fog(x)

Since range of g is not a subset of the domain of f i.e. $[-1, \infty) \not\subset [0, \infty)$

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is $\{x \in R$, the domain of g : $g(x) \in [0, \infty)$, the domain of f$\}$.

Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$

i.e. $D = \{x \in R: 0 \le x^2 - 1\} = \{x \in R: x \le -1$ or $x \ge 1\} = (-\infty, -1] \cup [1, \infty)$

$fog(x) = f\{g(x)\} = f(x^2 - 1) = \sqrt{x^2 - 1}$ Its range is $[0, \infty)$.

Inverse of a Function :

Let $y = f(x) : A \to B$ be a one-one and onto function. i.e. bijection, then there will always exist bijective function $x = g(y) : B \to A$ such that if (p, q) is an element of f, (q, p) will be an element of g and the functions f(x) and g(x) are said to be inverse of each other. g(x) is also denoted by $f^{-1}(x)$ and f(x) is denoted by $g^{-1}(x)$

Note :

(i) The inverse of a bijection is unique.

(ii) Inverse of an even function is not defined.

To find the inverse of f we follow the following steps.

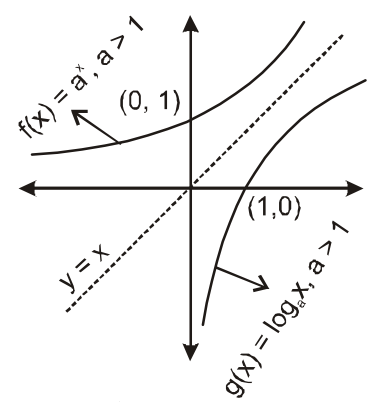
Step - I : put $f(x) = y$, where $y \in B$ and $x \in A$

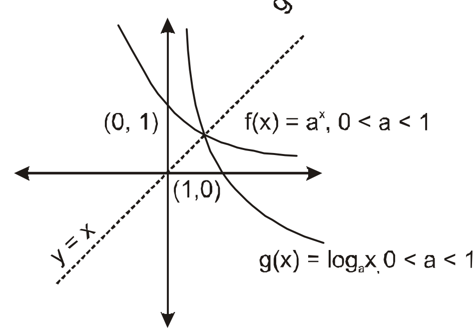
Setp - II : Solve $f(x) = y$ to obtain x in terms of y.

Setp- III : In the relation obtained in step III replace x by $f^1(y)$ to obtain the required inverse of f.

Properties of Inverse Function :

(a) The graphs of f and g are the mirror images of each other in the line $y = x$. For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line $y = x$ as shown below.





(b) Normally points of intersection of f and f^{-1} lie on the straight line $y = x$. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also. e.g $f(x) = 1/x$

(c) In general fog(x) and gof(x) are not equal. But if f and g are inverse of each other, then gof = fog. fog(x) and gof(x) can be equal even if f and g are not inverse of each other. e.g. $f(x) = x + 1$, $g(x) = x + 2$. However if $fog(x) = gof(x) = x$, then $g(x) = f^{-1}(x)$

(d) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

(e) If f(x) and g(x) are inverse function of each other, then $f'(g(x)) = \dfrac{1}{g'(x)}$

Solved Examples

Ex.50 If $f : R \to R$, $f(x) = 2x + 3$, then find $f^{-1}(x)$.

Sol. Since f is a bijection therefore its inverse mapping exists and $y = 2x + 3 \Rightarrow x = 2y + 3$

$\Rightarrow y = \dfrac{x-3}{2} \quad \therefore f^{-1}(x) = \dfrac{x-3}{2}$ **Ans.**

Ex.51 If $f : R \to R$ where $f(x) = x^2 + 3x + 7$ then find $f^{-1}(7)$.

Sol. Let $f^{-1}(7) = x \qquad \therefore f(x) = 7$

$\Rightarrow x^2 + 3x + 7 = 7 \qquad \Rightarrow x^2 + 3x = 0$

$\Rightarrow x = 0, -3$

$\therefore f^{-1}(7) = \{0, -3\}$ **Ans.**

Ex.52 If $f : R \to R$, $f(x) = x^3 + 2$ then find $f^{-1}(x)$.

Sol. $f(x) = x^3 + 2$, $x \in R$.

Since this is a one– one onto function therefore inverse of this function (f^{-1}) exists.

Let $f^{-1}(x) = y$

$\therefore x = f(y) \Rightarrow x = y^3 + 2 \Rightarrow y = (x-2)^{1/3}$

$\therefore f^{-1}(x) = (x-2)^{1/3}$. **Ans.**

Ex.53 (i) Determine whether $f(x) = \dfrac{2x+3}{4}$ for $f : R \to R$, is invertible or not? If so find it.

(ii) If $y = f(x) = x^2 - 3x + 1$, $x \geq 2$. Find the value of $g'(1)$ where g is inverse of f

Sol. (i) Given function is one-one and onto, therefore it is invertible.

$y = \dfrac{2x+3}{4} \Rightarrow x = \dfrac{4y-3}{2} \quad \therefore f^{-1}(x) = \dfrac{4x-3}{2}$

(ii) $y = 1 \Rightarrow x^2 - 3x + 1 = 1$

$\Rightarrow x(x-3) = 0 \Rightarrow x = 0, 3$

But $x \geq 2 \qquad \therefore x = 3$

Now $g(f(x)) = x$

Differentiating both sides w.r.t. x

$\Rightarrow g'(f(x)).\, f'(x) = 1 \Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}$

$\Rightarrow g'(f(3)) = \dfrac{1}{f'(3)} \Rightarrow g'(1) = \dfrac{1}{6-3} = \dfrac{1}{3}$

(As $f'(x) = 2x - 3$)

General tips :

If x, y are independent variables, then:

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x).\, f(y) \Rightarrow f(x) = x^n$, $n \in R$

(iii) $f(x + y) = f(x).\, f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.