THREE-DIMENSIONAL GEOMETRY

1.1 Basic concept

- **Distance formula.** Distance between two points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is

$AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

- **Section formula.** Coordinates of a point P, which divides the join of two given points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ in the ratio I : m **internally**, are $P\left(\frac{lx_2+mx_1}{l+m},\frac{ly_2+my_1}{l+m},\frac{lz_2+mz_1}{l+m}\right)$, and the coordinates of a point Q dividing the join in the ratio l: m **externally** are $Q\left(\frac{lx_2-mx_1}{l-m},\frac{ly_2-my_1}{l-m},\frac{lz_2-mz_1}{l-m}\right)$

- Coordinates of the mid-point P of the line segment joining the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ are $P\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$.

- **Direction cosines of a line :**

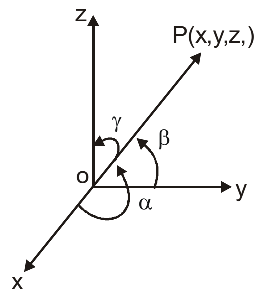
(i) The direction of line OP is determined by the angles α,β,γ which it makes with OX, OY and OZ respectively. These angles are called the direction angles and their cosines are called the direction cosines.

(ii) Direction cosines of a line are denoted by l, m, n,

$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

if we take the opposite direction of OP, then angles with axes are $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$

In this case $l=-\cos\alpha$, $m = -\cos\beta$, $n=-\cos\gamma$.



(iii) Sum of squares of direction cosines of a line is always 1.

$\ell^2 + m^2 + n^2=1$, i.e., $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

- **(i)** **Direction ratios of a line.** Numbers proportional to the direction cosines of a line are called direction ratios of a line. If a, b, c are direction ratios of a line then $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$.

(ii) If a, b, c are direction ratios of a line, then its direction cosines are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\pm\frac{b}{\sqrt{a^2+b^2+c^2}},\pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

(iii) Direction ratios of a line AB passing through the points $A(x_1,y_1, z_1)$ and $B(x_2,y_2, z_2)$ are x_2-x_1, y_2-y_1, z_2-z_1.

- **Projection of a line segment** on a given line. The projection of a line segment AB, where the coordinates of A and B are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively on the line whose direction cosines are l, m, n, is $(x_2-x_1)l + (y_2-y_1)m + (z_2-z_1)n$.

- **Angles between two lines.**

If θ is the angle between two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2, then,

(i) $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(ii) $\sin\theta = \pm\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}$

(iii) If the lines are parallel, then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$,

(iv) If the lines are perpendicular, then $l_1l_2 + m_1m_2 + n_1n_2 = 0$

1.2 Concept of line

- **Angle between two lines** whose direction ratios are a_1, b_1, c_1 and a_2, b_2, c_2, is

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + +b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(i) If lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (ii) If lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

- **Vector equation of a line** passing through a point with position vector $\vec{a}$ and along direction $\vec{m}$ is $\vec{r} = \vec{a} + \lambda\vec{m}, \lambda$ is a scalar (parameter)

- **Cartesian equations** (Equations in Symmetric form) of a line passing through point (x_1, y_1, z_1,) and having direction ratios a, b, c are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

- **Vector equation of a line** passing through two points. with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, λ is a scalar (parameter)

- **Equations of a line** passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$.

- **Angle θ between the two given lines** $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by cos θ = $\frac{\vec{b}_1\vec{b}_2}{\vec{b}_1 || \vec{b}_2}$

(i) If lines are perpendicular, then $\vec{b}_1\, \vec{b}_2 = 0$

(ii) If lines are parallel then $\vec{b}_1 \times \vec{b}_2 = \vec{0}$ or $\vec{b}_1 = t\vec{b}_2$ t is scalar (parameter)

- Angle θ between the two given lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(i) If lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$. (ii) If lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

- Shortest distance between two skew lines is the length of the line segment, which is perpendicular to the two given lines, If two given lines are $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, then shortest distance is

$$\left|\frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|.$$

If shortest distance is zero, then lines intersect and lines intersect in space if they are coplanar. Hence above lines are coplanar if $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$.

- **Shortest distance 'd' between the lines** $\ell_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\ell_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

Two lines intersect if shortest distance is zero. Lines intersect in space if they are coplanar.

Hence, if above lines l_1, l_2 intersect or are coplanar, then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- Distance between parallel lines. If two lines ℓ_1, ℓ_2 are parallel, then they are coplanar.

  Let the lines be $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$.

  The distance between n parallel lines is $\left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$.

1.3 Concept of plane

- General equation of a plane in vector form is $\vec{r}.\vec{n}+d=0$, $\vec{n}$ is a vector normal (perpendicular) to the plane.
- General equation of a plane in Cartesian form is ax+by+cz+d=0, where a, b, c, are direction ratios of normal (perpendicular) to the plane.
- General equation of a plane passing through a point with position vector $\vec{a}$ is $(\vec{r}-\vec{a}).\vec{n}=0$, where $\vec{n}$ is a vector perpendicular to the plane.
- General equation of a plane passing through a point (x_1, y_1, z_1) is :
  $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, a, b, c are direction ratios of a line perpendicular to the plane.
- Intercept form of equation of a plane. General equation of a plane which cuts off intercepts a, b and c on x-axis, y-axis and z-axis respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.
- Equation of a plane in normal form is $\vec{r}.\hat{n}$ = p, where $\hat{n}$ is a unit vector along perpendicular from origin and 'p' is distance of plane from origin. As 'p' is positive being distance, R.H.S. is always positive.
- Equation of a plane in normal form is lx + my+ nz = p, where l, m, n are direction cosines of perpendicular from origin and 'p' is distance of plane from origin. As 'p' is positive being distance, so R.H.S. is always positive.
- **Equation of a plane passing through three non-collinear points.** If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of three given non-collinear points, then equation of a plane through three points is given by

$$(\vec{r}-\vec{a})\ \{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=0.$$

- **In cartesian form, equation of a plane** passing through the points (x_1,y_1,z_1) and (x_3, y_3, z_3) is and (x_3, y_3, z_3) is $\begin{vmatrix} x-x1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$.

- **If θ angle between two planes** $\vec{r}.\vec{n}_1+d_1=0$ and $\vec{r}.\vec{n}_2+d_2=0$ then $\cos\theta = \dfrac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

  (i) If planes are perpendicular, then $\vec{n}_1.\vec{n}_2 = 0$.

  (ii) If planes are parallel, then $\vec{n}_1 \times \vec{n}_2 = \vec{0}$ $\vec{n}_1 = t\vec{n}_2$, t is a scalar (parameter).

- **If θ is angle between two planes** $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$

  Then $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$.

  If planes are perpendicular, then $a_1a_2+b_1b_2+c_1c_2=0$ and if planes are parallel then $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$.

- If θ is angle between line $\vec{r}=\vec{a}+\lambda\vec{m}$ and the plane $\vec{r}.\vec{n}+d=0$, then $\sin\theta = \dfrac{\vec{m}.\vec{n}}{|\vec{m}||\vec{n}|}$.

  If line is parallel to plane, then $\vec{m}.\vec{n}=0$ and if line is perpendicular to plane, the $\vec{m}\times\vec{n}=\vec{0}$ or $\vec{m}=t\vec{n}$, t is scalar (parameter).

- **If θ is angle between line** $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{B_1}=\dfrac{Z-Z_1}{c_1}$ and the plane ax + by + cz + d = 0

  then $\sin\theta = \dfrac{aa_1+bb_1+cc_1}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}$.

  If line is parallel to the plane, then $aa_1 + bb_1 + cc_1 = 0$ and, if line is perpendicular to the plane, the $\dfrac{a}{a_1}=\dfrac{b}{b_1}=\dfrac{c}{c_1}$.

- **General equation of a plane parallel to the plane** $\vec{r}.\vec{n}+d=0$ is $\vec{r}.\vec{n}+\lambda=0$, where λ is a constant (parameter) and can be calculated from a given condition.
- **General equation of a plane parallel to the plane** ax +by+cz+d=0 is ax +by+cz+λ =0, where λ is a constant (parameter) and can be calculated from a given condition.
- **General equation of a plane** passing through the line of the intersection of planes $\vec{r}.\vec{n}_1+d_1=0$ and $\vec{r}.\vec{n}_2+d_2=0$ is $\vec{r}.(\vec{n}_1+\lambda\vec{n}_2)$ + $(d_1 + \lambda d_2) = 0$, where λ is a constant (parameter) and can be calculated from a given condition.
- **General equation of a plane** passing through the intersection of planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is $a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0$ where λ is a constant (parameter and can be calculated) from a given condition.

- **Distance of a plane** $\vec{r}.\vec{n}+d=0$, from a point with position vector $\vec{a}$, is $\left|\dfrac{\vec{a}.\vec{n}+d}{|\vec{n}|}\right|$.

- **Distance of a plane** ax+by+cz+d=0, from a point (x_1,y_1,z_1), is $\left|\dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$.

SOLVED PROBLEMS

Ex.1 ***Find the direction cosines of y-axis.***

Sol. y-axis makes 90^0, 0^0, 90^0, angles with x, y and z axis respectively.

$\therefore$ Direction Cosines are cos 90^0, cos 0^0, cos 90^0 i.e., 0, 1, 0.

Ex.2 ***Find the direction ratios of the line***

$$\frac{x+2}{2} = \frac{2y-1}{3} = \frac{3-z}{5}.$$

Sol. Line is $\frac{x+2}{1} = \frac{2\left(y-\frac{1}{2}\right)}{3} = \frac{-(z-3)}{5} \Rightarrow \frac{x+2}{2} = \frac{y-\frac{1}{2}}{3} = \frac{z-3}{-10}$

Direction ratios are 2, 3, −10.

Ex.3 ***Find the angle between the line***

$\vec{r} = \left(2\hat{i} - \hat{j} + 3\hat{k}\right) + \lambda\left(3\hat{i} - \hat{j} + 2\hat{k}\right)$ ***and the plane***

$\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 3.$

Sol. $\sin\theta = \frac{\left(3\hat{i} - \hat{j} + 2\hat{k}\right).\left(\hat{i} + \hat{j} + \hat{k}\right)}{\sqrt{9+1+4}\sqrt{1+1+1}} = \frac{3-1+2}{\sqrt{14}\sqrt{3}} = \frac{4}{\sqrt{42}}$

Ex.4 ***Find the direction cosines of the two lines which are connected by the relations,*** ***$\ell - 5m + 3n = 0$ and $7\ell^2 + 5m^2 - 3n^2 = 0$.***

Sol. The given equation are $\ell - 5m + 3n = 0$ ...(i)

$7l^2 + 5m^2 - 3n^2 = 0$ ...(ii)

From (i), we have $l = 5m - 3n$. substituting $l = 5m - 3n$ in (ii), we get

$7(5m-3n)^2+5m^2-3n^2 = 0 \Rightarrow 6m^2 - 7mn + 2n^2 = 0$

$\Rightarrow 6m^2-3mn-4mn+2n^2 = 0$

$\Rightarrow (3m-2n)(2m-n)=0 \Rightarrow m = \frac{2}{3}n$ or $m = \frac{n}{2}$

If $m = \frac{2}{3}n$, then from (i), we obtain $l = \frac{1}{3}n$

If $m = \frac{n}{2}$, then from (i), we obtain $l = -\frac{n}{2}$

Thus, direction ratios of two lines are

$\frac{n}{3}, \frac{2}{3}n, n$ and $\frac{-n}{2}, \frac{n}{2}, n$, i.e., 1,2,3, and −1,1,2

Hence, their direction cosines are

$\pm\frac{1}{\sqrt{14}} \pm \frac{2}{\sqrt{14}}, \pm\frac{3}{\sqrt{14}}$ and $\pm\frac{-1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{2}{\sqrt{6}}$

Ex.5 ***Find the direction cosines of the line $\frac{x+2}{4} = \frac{2y-7}{6}$, z =-5. Also, find the vector equation of the line through the point A (-1, 2,3,) and parallel to the given line.***

Sol. The equation or the line is given as

$$\frac{x+2}{4} = \frac{2y-7}{6} = -5$$

This can be rewritten as $\frac{x+2}{4} = \frac{y-\frac{7}{2}}{3} = \frac{z+5}{0}$

$\therefore$ Its direction cosines are $\langle 4,3,0 \rangle$

Hence, its direction cosines are $\left\langle \frac{4}{5}, \frac{3}{5}, 0 \right\rangle$

[Dividing by $\sqrt{4^2+3^2+0^2}$, i.e., by 5]

The vector equation of a line passing through the point (−1, 2, 3) and parallel to the given line is

$\vec{r} = -\hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(4\hat{i} + 3\hat{j}\right), \lambda$ being a scalar.

Ex.6 ***Find the angle between the following pairs of lines:***

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)$ ***and***

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda\left(\hat{i} - \hat{j} - 2\hat{k}\right)$ ***and***

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

Sol. We know that if $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ be two lines, the acute angle θ between these lines is given by

$$\cos\theta = \left|\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

(i) Here $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore \quad |\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

and $\quad |\vec{b}_2| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

$\vec{b}_1.\vec{b}_2 = (3)(1) + (2)(2) + (6)(2) = 3 + 4 + 12 = 19$

Thus, $\cos\theta \left|\frac{19}{7\times 3}\right| = \frac{19}{21}$

Hence, the required angle between the given lines is $\cos^{-1}\left(\frac{19}{21}\right)$.

(ii) Here $\vec{b} = \hat{i} - \hat{j} - 2k$ and $\vec{b}_2 = 3i - 5j - 4k$

$\therefore \quad \left|\vec{b}_1\right| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

$\left|\vec{b}_2\right| = \sqrt{3^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$ and

$\vec{b}_1.\vec{b}_2 = (1)(3) + (-1)(-5) + (-2)(-4) = 3+5+8=16$

Hence, $\cos\theta = \left|\frac{16}{\sqrt{6}\times 5\sqrt{2}}\right| = \frac{16}{5\sqrt{12}} = \frac{8}{5\sqrt{3}}$

Thus, the required angle between the given lines is $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$

Ex.7 Find the angle between the following pair of lines :

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ ***and*** $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z+5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ ***and*** $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Sol. (i) The d.r.'s. of the first line are $\langle 2,5,-3\rangle$ and the d.r.'s of the second line are $\langle -1,8,4\rangle$. if θ is the acute angle between the two lines, then

$$\cos\theta = \left|\frac{2\times(-1) + 5\times 8 + (-3)\times 4}{\sqrt{2^2 + 5^2 + (-3)^2}\sqrt{(-1)^2 + 8^2 + 4^2}}\right|$$

$$= \left|\frac{-2+40-12}{\sqrt{38}\sqrt{81}}\right| = \frac{26}{9\sqrt{38}}$$

Hence, the required angle is $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

(ii) The d.r.'s of the first line are $\langle 2,2,1\rangle$ and the d.r.'s of the second line are $\langle 4,1,8\rangle$. If θ is the acute between the two lines, then

$$\cos\theta = \left|\frac{2\times 4 + 2\times 1 + 1\times 8}{\sqrt{2^2+2^2+1}\sqrt{4^2+1^2+8^2}}\right| = \left|\frac{8+2+8}{\sqrt{9}\sqrt{81}}\right| = \frac{18}{3\times 9} = \frac{2}{3}$$

Hence, the required angle is $\cos^{-1}\left(\frac{2}{3}\right)$.

Ex.8 Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ ***and*** $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ ***are perpendicular to each other.***

Sol. The d.r.'s of the first line are (7, −5, 1) and the d.r's of the second line are $\langle 1,2,3\rangle$. The two lines are perpendicular if (7) (1)+(-5) (2)+(1) (3)=0

i.e., 7 - 10 + 3 = 0, which is true.

Hence, the given lines are perpendicular.

Ex.9 Find the coordinates of the point where the line through the line through the points A(3, 4, 1) and B (5, 1, 6) crosses the XY-plane.

Sol. The vector equation of the line AB through the points A (3, 4, 1) and B (5, 1, 6) is

$\vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda\left[(5-3)\hat{i} + (1-4)\hat{j} + (6-1)\hat{k}\right]$

i.e., $\quad \vec{r} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda\left(2\hat{i} - 3\hat{j} + 5\hat{k}\right)$ ...(i)

Let P be the point where the line AB crosses the XY-plane. Then the position vector of the point P is of the form $x\hat{i} + y\hat{j}$. This point must satisfy Eq. (1). So,

$x\hat{i} + y\hat{j} = 3\hat{i} + 4\hat{j} + \hat{k} + \lambda\left(2\hat{i} - 3\hat{j} + 5\hat{k}\right)$

Equating the coefficient of base vectors, we have

$\Rightarrow \quad x = 3 + 2\lambda,\ \ y = 4 - 3\lambda$ and $0 = 1 + 5\lambda$

$\lambda = -\frac{1}{5}$ and thus, $\quad x = \frac{13}{5}$ and $y = \frac{23}{5}$

Hence, the coordinates of the required point are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Ex.10 Prove that the lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$

Sol. The given equations can be rewritten as

$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c} \qquad ....(1) \qquad \text{and}$$

$$\frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'} \qquad ....(2)$$

Lines (1) and (2) will be perpendicular

if $aa' + 1.1 + cc' = 0$ i.e. $aa' + cc' + 1 = 0$. Hence proved.

Ex.11 Aline passing through (2,-1,3) and is perpendicular to the lines

$$\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda\left(2\hat{i} - 2\hat{j} + \hat{k}\right) \quad \text{and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - 3\hat{k} + \mu\left(i + 2\hat{j} + 2\hat{k}\right).$$

Obtain its vector equation.

Sol. The required line passes through (2.-1-3) and is perpendicular to the lines which are parallel to vectors $\vec{b}_1 = 2\hat{i} - 2\hat{j} + k$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively. So, it is parallel to the vector $\vec{b}_1 \times \vec{b}_2$.

Now, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = (-4-2)\hat{i} + (1-4)\hat{j} + (4+2)\hat{k}$

$= -6\hat{i} - 3\hat{j} + 6\hat{k}$

Thus, the required equation of the line is

$\vec{r} = \left(2\hat{i} - \hat{j} + 3\hat{k}\right) + \lambda\left(-6\hat{i} - 3\hat{j} + 6\hat{k}\right)$

Ex.12 The point A (4,5,10), B (2,3,4) and C (1,2,–1) are three vertices of parallelogram ABCD. Find vector equation for the sides AB, BC and also find the coordinates of D.

Sol. Let the coordinates of D be (α, β, γ), Then the p.v. of A, B, C, D referred to the origin are respectively.

$\vec{a} = 4\hat{i} + 5\hat{j} + 10\hat{k} \qquad \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{c} = \hat{i} + 2\hat{j} - \hat{k} \qquad \vec{d} = \alpha\hat{i} + \beta\hat{j} - \gamma\hat{k}$

(i) The vector equation of the line AB is

$\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$

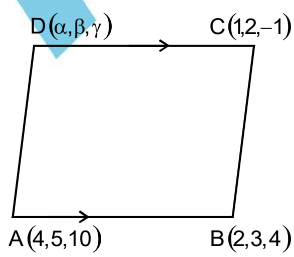
$= \left(4\hat{i} + 5\hat{j} + 10\hat{k}\right) + \lambda\left(-2\hat{i} - 2\hat{j} - 6\hat{k}\right)$

(ii) The vector equation of the line BC is

$\vec{r} = \vec{b} + \mu\left(\vec{c} - \vec{b}\right) = 2\hat{i} + 3\hat{j} + 4\hat{k} + \mu\left(-\hat{i} - \hat{j} - 5\hat{k}\right)$

(iii) Since ABCD is a parallelogram,

$\vec{AB} = \vec{DC}$



$\Rightarrow \vec{b} - \vec{a} = \vec{c} - \vec{d}$

$\Rightarrow \vec{d} = \vec{c} - \left(\vec{b} - \vec{a}\right)$

$= \left(\hat{i} + 2\hat{j} - \hat{k}\right) - \left(-2\hat{i} - 2\hat{j} - 6\hat{k}\right)$

$= 3\hat{i} + 4\hat{j} + 5\hat{k}$

Hence, the coordinates of D are (3, 4, 5).

Ex.13 Find the shortest distance between the following lines : $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

Sol. Comparing the given equation with

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

respectively, we have

$(x_1, y_1, z_1) = (-1, -1, -1); \quad (a_1, b_1, c_1) = (7, -6, 1)$

$(x_2, y_2, z_2) = (3, 5, 7); \quad (a_2, b_2, c_2) = (1, -2, 1)$

The shortest distance (S.D.) between the given lines is given by

$$SD = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right|$$

$$= \left| \frac{\begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}}{\sqrt{(-6+2)^2 + (1-7)^2 + (-14+6)^2}} \right|$$

$$= \left| \frac{4(-6+2)+6(1-7)+8(-14+6)}{\sqrt{16+36+64}} \right|$$

$$= \left| \frac{-16-36-64}{\sqrt{116}} \right| = \left| \frac{-116}{2\sqrt{29}} \right| = 2\sqrt{29} \text{ units}$$

Alternative Method

The equation of two given lines are

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \quad \text{......(1)} \quad \text{and}$$

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad \text{......(2)}$$

The given lines have direction ratios <7, -16, 1> and <1, -2, 1>. So, they are parallel to the vectors

$$\vec{b}_1 = 7\hat{i} - 6\hat{j} + \hat{k} \text{ and } \vec{b}_2 = \hat{i} - 2\hat{j} + \hat{k}$$

The given lines pass through the points (-1, -1, -1) and (3, 5, 7) respectively. So, the p.v. of these points are

$$\vec{a}_1 = -\hat{i} - \hat{j} - \hat{k} \text{ and } \vec{a}_2 = 3\hat{i} + 5\hat{j} + 7\hat{k}$$

Now, $\vec{a}_2 - \vec{a}_1 = 4\hat{i} + 6\hat{j} + 8\hat{k}$ and $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$$= (-6+2)\hat{i} + (1-7)\hat{j} + (-14+6)\hat{k} \quad = -1\hat{i} - 6\hat{j} - 8\hat{k}$$

So, $\left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-4)^2 + (-6)^2 + (-8)^2}$

$$= \sqrt{16+36+64} = \sqrt{116} = 2\sqrt{29}$$

Hence, the shortest distance (S.D.) between the lines is given by

$$\text{S.D.} = \left| \frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$= \left| \frac{(-4\hat{i} - 6\hat{j} - 8\hat{k}).(4\hat{i} + 6\hat{j} + 8\hat{k})}{2\sqrt{29}} \right|$$

$$= \left| \frac{-16-36-64}{2\sqrt{29}} \right| = \left| \frac{-116}{2\sqrt{29}} \right| = 2\sqrt{29} \text{ units}$$

Note : Find the shortest distance between the given lines by vector method (even if equations given are in cartesian form).

Ex.14 Find the distance between the parallel lines whose vector equations are

$$\vec{r} = \hat{i} + \hat{j} + \lambda\left(2\hat{i} - \hat{j} + \hat{k}\right) \text{ and}$$

$$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu\left(4\hat{i} - 2\hat{j} + 2\hat{k}\right)$$

Sol. The given lines are $\vec{r} = \hat{i} + \hat{j} + \lambda\left(2\hat{i} - \hat{j} + \hat{k}\right)$

and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu\left(4\hat{i} - 2\hat{j} + 2\hat{k}\right)$

$$= 2\hat{i} + \hat{j} - \hat{k} + 2\mu\left(2\hat{i} - \hat{j} + \hat{k}\right) \quad = 2\hat{i} + \hat{j} - \hat{k} + v\left(2\hat{i} - \hat{j} + \hat{k}\right),$$

where $v = 2\mu$

These two lines pass through the point having p.v. $\vec{a}_1 = \hat{i} + \hat{j}; \vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k}$ respectively. Both these lines are parallel to the vector $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$. Hence, the distance (d) between the two given parallel lines is given by

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

We have $\vec{b} \times (\vec{a}_2 - \vec{a}_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$

$$= (1-0)\hat{i} + (1+2)\hat{j} + (0+1)\hat{k} \quad = \hat{i} + 3\hat{j} + \hat{k}$$

$$\left|\vec{b} \times (\vec{a}_2 - \vec{a}_2)\right| = \sqrt{1^2 + 3^2 + 1^2} \quad = \sqrt{1+9+1} = \sqrt{11}$$

$$|\vec{b}| = \sqrt{2^2 + (-1)^2 + 1^2} \quad = \sqrt{4+1+1} = \sqrt{6}$$

Putting these values in (1), we has $d = \frac{\sqrt{11}}{\sqrt{6}} = \sqrt{\frac{11}{6}}$ units

Ex.15 Find the intercepts cut off by the plane $2x + y - z = 5$ on the axes.

Sol. The given equation of the plane is

$$2x + y - z = 5 \Rightarrow \frac{2}{5}x + \frac{1}{5}y - \frac{1}{5}z = 1$$

$$\Rightarrow \quad \frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1$$

Hence, the intercepts on the axes are $\frac{5}{2}$, 5 and -5.

Ex.16 If O be the origin and the coordinates of P be (1, 2, -3), then find the equation of the plane passing through P and perpendicular to OP.

Sol. The position vector of the point P is $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{N} = \overrightarrow{OP} = \hat{i} + 2\hat{j} - 3\hat{k}$

Hence, the vector equation of the plane passing through P and perpendicular to OP is given by $(\vec{r} - \vec{a}).\vec{N} = 0$ or $\vec{r}.\vec{N} = \vec{a}.\vec{N}$

$$\Rightarrow \quad \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) = (\hat{i} + 2\hat{j} - 3\hat{k}).(\hat{i} + 2\hat{j} - 3\hat{k})$$

$$\Rightarrow \quad \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) = 1 + 4 + 9 = 14$$

which is the required vector equation of the plane.

substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (1), we have

$(x\hat{i} + y\hat{j} + z\hat{k}).\ (\hat{i} + 2\hat{j} - 3\hat{k}) = 14 \quad \Rightarrow \quad x + 2y - 3z = 14$

which is the cartesian equation of the plane.

Ex.17 Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2$.

Sol. The given plane can be written as

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + \hat{j} + \hat{k}) = 2 \quad \Rightarrow \quad x + y + z - 2 = 0$

Any plane parallel to (1) is $x + y + z - \hat{k} = 0$

Since it passes through the point (a, b, c)

$a + b + c - \hat{k} = 0$ i.e., $\hat{k} = a + b + c$

Substituting this value of k in (2),

we have $x + y + z = a + b + c$

which is the required equation of the plane.

The vector equation of the plane is

$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = a + b + c$

Ex.18 Find the vector equation of the line passing through (1,2,3) and perpendicular to the plane $\vec{r}.(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$.

Sol. Since the required line passes through the point A(1, 2, 3) with position vector $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and is perpendicular to the plane.

$\vec{r}.(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$, the vector $\hat{i} + 2\hat{j} - 5\hat{k}$ is along a normal to the given plane. Therefore, the required line is along the direction of the vector $\hat{i} + 2\hat{j} - 5\hat{k}$. Hence, the equation of the line is

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda.(\hat{i} + 2\hat{j} - 5\hat{k})$

Ex.19 Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}.(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}.(3\hat{i} + \hat{j} + \hat{k}) = 6$.

Sol. Let the required equation of the line passing through the point (1,2,3) be

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$ ......(1)

The normal vectors of the given two planes are $\hat{i} - \hat{j} + 2\hat{k}$ and $3\hat{i} + \hat{j} + \hat{k}$

Since the required line is $\perp$ar to the given planes, the vector $a\hat{i} + b\hat{j} + c\hat{k}$ is $\perp$ar to each of the normal vectors.

$$\therefore \quad (a\hat{i} + b\hat{j} + c\hat{k}).(\hat{i} - \hat{j} + 2\hat{k}) = 0$$

i.e., $a - b + 2c = 0$ and

$(a\hat{i} + b\hat{j} + \hat{k}).(3\hat{i} + \hat{j} + \hat{k}) = 0$

i.e., $3a + b + c = 0 \Rightarrow \frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3} = k$ (say)

$\Rightarrow \quad a = -3\hat{k},\ b = 5\hat{k}$ and $c = 4\hat{k}$

From (1),we have $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \mu(-3\hat{i} + 5\hat{j} + 4\hat{k})$

where $\mu = \lambda\hat{k}$

which is the required equation of the line.

Ex.20 ***Find the cartesian equation of the plane passing through the points (0, 0, 0) and (3, -1, 2) and parallel to the line $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$.***

Sol. The general equation of a plane is

$ax+by+cz+d=0$ ....(1)

Since it passes through (0, 0, 0) and (3, -1, 2)

$a.0+b.0+c.0+d=0$ i.e., d=0

$3a-b+2c+d=0$ i.e., 3a-b+2c=0 ....(2)

If (1) is parallel to the given line, then normal to it is perpendicular to the given line.

so, (1) (a) - (4) (b) + (7) (c) = 0

$\Rightarrow$ a − 4b + 7c = 0 ....(3)

Solving (2) and (3) by cross multiplication, we get

$\Rightarrow \frac{a}{1}=\frac{b}{-19}=\frac{c}{-11}=\lambda \Rightarrow a=\lambda,\ b=-19\lambda,\ c=-11\lambda$

Hence, the required plane is $\lambda x-19\lambda y-11\lambda z=0$

i.e. $x-10y-11z=0$

which is the cartesian equation of the plane.

Ex.21 ***If the points (1, 1, p) and (-3, 0, 1) be equidistant from the plane $\vec{r}.\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$, then find the value of p.***

Sol. let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A (1,1, p) and B (-3, 0, 1) respectively. Then,

$a=\hat{i}+\hat{j}+p\hat{k}$ and $\vec{b}=-3\hat{i}+\hat{k}$

Distance of A from the plane

$$=\frac{\left|\left(\hat{i}+\hat{j}+p\hat{k}\right).\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13\right|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$=\frac{|3+4-12p+13|}{\sqrt{9+16+144}}=\frac{|20-12p|}{13}$$

Distance of B from the plane

$$=\frac{\left|\left(-3\hat{i}+\hat{k}\right).\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13\right|}{\sqrt{3^2+4^2+(-12)^2}}=\frac{|-9+0-12+13|}{\sqrt{9+16+144}}=\frac{8}{13}$$

Equating the two distances,

we have $|20-12p|=8$ $p=1$ and $p=\frac{7}{3}$

Ex.22 ***Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$ and parallel to x-axis.***

Sol. Any plane passing through the intersection of the given planes is

$\vec{r}.\left[\left(\hat{i}+\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}-\hat{k}\right)\right]=1-4\lambda$

$\Rightarrow$ $\vec{r}.\left[(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1-\lambda)\hat{k}\right]=1-4\lambda$

The d.c.'s of x-axis are <1, 0, 0>

Since the plane (1) is parallel to x-axis, the normal of the plane is perpendicular to x-axis.

$\therefore$ (1) $(1+2\lambda)+0(1+3\lambda)+0(1-\lambda)=0$

$\Rightarrow$ $1+2\lambda=0$ $\Rightarrow$ $\lambda=-\frac{1}{2}$

Hence, the equation of the plane is

$\vec{r}.\left(-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}\right)=3$ $\vec{r}.\left(-\hat{j}+3\hat{k}\right)=6$

Ex.23 ***Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0$, $\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$, and which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$.***

Sol. Any plane passing through the intersection of the given planes is $\vec{r}.[(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+\hat{j}-\hat{k})]=4-5\lambda$

i.e., $\vec{r}.[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}]=4-5\lambda$.

The d.r.'s of the normal of the plane are $<5, 3, -6>$

Since plane (1) is perpendicular to the plane

$\vec{r}.(5i+3j-6k)+8=0$, $(5)(1+2\lambda)+3(2+\lambda)-6(3-\lambda)=0$

$\Rightarrow \quad 5+10\lambda+6+3\lambda-18+6\lambda=0$

$\Rightarrow \quad 19\lambda=7$ i.e., $\lambda\frac{7}{19}$

Hence, the required equation of the plane is $\vec{r}.(33\hat{i}+45\hat{j}+50\hat{k})=41$

In cartesian from, the equation of the plane is $33x+45y+50z=41$

Ex.24 Find the distance between the two parallel planes : $2x+3y+4z=4$ and $4x+6y+8z=12$

Sol. The vector equations of the given planes are $\vec{r}.(2\hat{i}+3\hat{j}+4\hat{k})=4$ and $\vec{r}.(4\hat{i}+6\hat{j}+8\hat{k})=12$

Let $\vec{a}$ be the p.v. of any point on the plane $\vec{r}.(2\hat{i}+3\hat{j}+4\hat{k})=4$. Then $\vec{a}.(2\hat{i}+3\hat{j}+4\hat{k})=4$

$\Rightarrow \quad \vec{a}.(4\hat{i}+6\hat{j}+8\hat{k})=8$

Now, the length of the perpendicular from $\vec{a}$ to $\vec{r}.(4\hat{i}+6\hat{j}+8\hat{k})=12$ is

$$=\frac{|\vec{a}.(4\hat{i}+6\hat{j}+8\hat{k})-12|}{\sqrt{4^2+6^2+8^2}}=\frac{|8-12|}{\sqrt{16+36+64}}=\frac{4}{116}=\frac{2}{\sqrt{29}}\text{units}$$

Ex.25 Find the point where the line $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+3}{4}$ meets the plane $2x+4y-z=1$.

Sol. The general point on the given line is $(2\lambda+1, -3\lambda+2,\ 4\lambda-3)$

If this point is to lie on the given plane $2x+4y-z=1$, then

$2(2\lambda+1)+4(-3\lambda+2)-(4\lambda-3)=1$

$\Rightarrow 4\lambda+2-12\lambda+8-4\lambda+3=1 \Rightarrow 12\lambda=12 \Rightarrow \lambda=1$

Hence the required point is

$(2\times1+1, -3\times1+2, 4\times1-3)$, i.e., (3, -1, 1).

Ex.26 Find the distance of the of point (2, 3, 4) measured along the line $\frac{x-4}{3}=\frac{y+5}{6}=\frac{z+1}{2}$ form the plane $3x+2y+2z+5=0$.

Sol. Equation of the line through (2, 3, 4) and parallel to the given line is $\frac{x-2}{3}=\frac{y-3}{6}=\frac{z-4}{2}$

Any point on this line is $(3\lambda+2, 6\lambda+3, 2\lambda+4)$.

If this point ,lies on the given plane, then

$3(3\lambda+2)+2(6\lambda+3)+2(2\lambda+4)+5=0$

$\Rightarrow \quad 25\lambda+25=0 \quad \Rightarrow \quad \lambda=-1$

EXERCISE – I | UNSOLVED PROBLEMS

Thus the point where the line meets the plane is (−1, −3, 2).

The required distance = Distance between (2, 3, 4) and (−1, −3, 2)

$= \sqrt{(2+1)^2+(3+3)^2+(4-2)^2} = \sqrt{9+36+4} = 7$ units.

Q.1 Find the value of k, so that the line $\vec{r} = (2-3\lambda)\hat{i} + (1+k\lambda)\hat{j} + \lambda\hat{k}$ is perpendicular to the line $\vec{r} = (1+k\lambda+2\lambda)\hat{i} + (3-2\lambda)\hat{j} + (1+2\hat{i})\hat{k}$.

Q.2 Find the point where the line joining the point (1,2,3) and (3,-1,2) intersects the XY-plane.

Q.3 Find the point of intersection of the lines

$\vec{r} = (\lambda-1)\hat{i} + (-3\lambda-2)\hat{j} + (-2\lambda-3)\hat{k}$ and $\vec{r} = (\mu-3)\hat{i} + (2\mu-1)\hat{j} + (\mu-2)\hat{k}$.

Q.4 Find the foot of the perpendicular drawn from the origin to the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{1-z}{2}$. Also, find the image of the origin in this line.

Q.5 Find the distance of the line $\frac{2x-1}{2} = \frac{y+1}{3} = \frac{z-2}{2}$ from the origin.

Q.6 Show that the lines $\vec{r} = (3\hat{i}+\hat{j}+2\hat{k}) + \lambda(4\hat{i}-\hat{j}-\hat{k})$ and $\vec{r} = (-\hat{i}+2\hat{j}+\hat{k}) + \mu(3\hat{i}-3\hat{j})$ intersect.

Q.7 Show that the lines $\vec{r} = (\hat{i}+\hat{j}+\hat{k}) + \lambda(2\hat{i}-\hat{j}-2\hat{k})$ and $\vec{r} = (2\hat{i}-\hat{j}+\hat{k}) + \mu(\hat{i}-\hat{j}+2\hat{k})$ are skew lines.

Q.8 Show that the lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-1}{2}$ and $\frac{x-3}{3} = \frac{y+2}{-2} = \frac{z-3}{3}$ are coplanar.

Q.9 Find the shortest distance between the lines $\frac{x-1}{2} = \frac{1-y}{3}, z = 2$ and $2x-1 = 3y+1 = z$.

Q.10 Find the shortest distance between the lines $\vec{r} = (2\hat{j}+3\hat{k}) + \mu(\hat{i}+2\hat{k})$ and $\vec{r} = (\hat{i}+\hat{k}) + \lambda(2\hat{j}-\hat{k})$

Q.11 Find the coordinates of the foot of the perpendicular drawn from the point (1,2,1) to the line joining (1,4,6) and (5, 4, 4).

Q.12 Find the equation of a line passing through (2,-1,-3) and perpendicular to the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x+1}{2} = \frac{y+2}{2} = \frac{z-1}{1}$.

Q.13 Find the equation of a line passing through the point (1,2,-3) and perpendicular to the line $\frac{x+1}{2} = \frac{y-3}{-2} = \frac{z}{-1}$.

Q.14 A plane meets the coordinate axes in A, B and C. If the centroid of ΔABC is (1,2,-2), find the equation of the plane.

Q.15 Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{2}$ and $\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{2}$.

Q.16 If $2\hat{i}-\hat{j}+\hat{k}$ is normal vector to a plane and the distance of the plane from the origin is $5\sqrt{6}$, find the equation of the plane both in vector and Cartesian form.

Q.17 Find the distance of the plane passing through the points (1,1,2,), (2,1,-1) and (-1,1,-2) from the origin.

Q.18 Find the point of intersection of the line passing through the points (−1, 2, −1) and (3,1,3) with the plane 2x − y + z − 2 = 0.

Q.19 Find the point if intersection of the line $\vec{r}=(2\hat{i}+2\hat{j}+\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and the plane $\vec{r}.(\hat{i}-\hat{j}-\hat{k})=5$.

Q.20 Find the equations of the line of intersection of the planes $2x+y-z-1=0$ and $x+2y+z-2=0$.

Q.21 Find the image of the point (−2, −1, 3) in the plane $\vec{r}.(\hat{i}+2\hat{j}-\hat{k})=2$.

Q.22 Find the equation of a plane passing through the points (1,2,-1), (2,1,2) and (3,-1,1).

Q.23 Find the equation of a plane passing through the origin and perpendicular to the planes $\vec{r}.(\hat{i}-2\hat{j}+\hat{k})=3$ and $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})=5$.

Q.24 Find the equation of a plane passing through the point (2,-1,1) and parallel to the lines $\frac{x-2}{3}=\frac{y-3}{2}=z$ and $\frac{x+1}{2}=\frac{y+2}{3}=\frac{z+3}{4}$.

Q.25 Find the equation of a plane passing through the point (3,-1,2) and passing though the intersection of the planes x+y-2z+1=0 and 2x-y+3z-5=0.

Q.26 Find the equation of a plane containing the line of intersection of the plane x − y − z − 1 = 0 and x + y + 2z − 4 = 0, and perpendicular to the plane 2x-y-3z-5=0.

Q.27 Find the equation of a plane containing the point (2,-1,-3) and the line $\frac{x-1}{2}=\frac{y+1}{-2}=\frac{z-2}{3}$.

Q.28 Find the equation of a plane containing the lines $\frac{x-1}{3}=\frac{y-1}{1}, z=-1$ and $\frac{x-4}{2}=\frac{z+1}{3}, y=0$.

Q.29 Find the equation of a plane containing the points (3,4,2) and (7,0,6), and perpendicular to the plane 2x − 5y = 15.

Q.30 Find the vector equation of the following plane in the scalar-product form : $\vec{r}=(\hat{i}-\hat{j})+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-2\hat{j}+3\hat{k})$.

EXERCISE – II   BOARD PROBLEMS

Q.1 Find the vector equation of a line passing through the point, whose position vector is and perpendicular to the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}+5\hat{k})+2=0$. Also find the point of intersection of this line and the plane.

Q.2 Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i}-7\hat{j}+4\hat{k})=3$ and $\vec{r}\cdot(3\hat{i}-5\hat{j}+4\hat{k})+11=0$ and passing through the point (–2, 1, 3).

Q.3 Show that the line L, whose vector equation is $\vec{r}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}-\hat{j}+4\hat{k})$ parallel to the plane π, whose vector equation is $\vec{r}.(\hat{i}+5\hat{j}+\hat{k})=5$ and find the distance between them.

Q.4 Find the distance of the point (2, 3, 4) from the plane 3x + 2y + 2z + 5 = 0 measured parallel to the line $\frac{x+3}{3}=\frac{y-2}{6}=\frac{z}{2}$.

Q.5 Find the vector equation of the line passing through the point A(2, –1, 1) and parallel to the line joining the points B(–1, 4, 1) and C(1, 2, 2). Also find the cartesian equations of the line.

Q.6 Show that the lines $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ and $\frac{x-2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ do not intersect each other.

Q.7 Find the shortest distance between the lines whose vector equations are
$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$ and $\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}+(2s+1)\hat{k}$

Q.8 Find the equation of the plane passing through the point (1, 1, 1) and perpendicular to each of the following planes x + 2y + 3z = 7 and 2x – 3y + 4z = 0.

Q.9 Find the foot of the perpendicular from the point (0,2,3) on the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.

Q.10 Find the equation of the plane through the point (1, 1, 1) and perpendicular to each of the following planes :
x + 2y + 3z = 7 and 2x – 3y + 4z = 0.

Q.11 Find the length of the perpendicular drawn from the point (2, 3, 7) to the plane 3x – y – z = 7. Also find the coordinates of the foot of the perpendicular.

Q.12 Find the point where the line joining the points (1, 3, 4) and (–3, 5, 2) intersects the plane $\vec{r}.(2\hat{i}+\hat{j}+\hat{k})+3=0$. Is this point equidistant from the given points ?

Q.13 Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect.
Also find the point of intersection.

Q.14 The vector equation of two lines are $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$. Find the shortest distance between the above lines.

Q.15 Find the equation of the plane passing through the line of intersection of the planes x − 2y + z = 1 and 2x + y + z = 8 and parallel to the line with direction ratios, 1, 2, 1. Also find the perpendicular distance of the point P(1, 3, 2) from this plane.

Q.16 Find the vector and Cartesian forms of the equation of the plane containing the lines

$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(-2\hat{i} + 3\hat{j} + 8\hat{k})$

Q.17 Find the equation of the line passing through the point P(4, 6, 2) and the point of intersection of the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7}$ and plane x + y − z = 8.

Q.18 Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i} + \hat{j} + 3\hat{k}) = 7$. $\vec{r}\cdot(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and the point (2, 1, 3).

Q.19 Find the equation of the plane which is perpendicular to the plane 5x + 3y + 6z + 8 = 0 and which contains the line of intersection of the planes x + 2y + 3z − 4 = 0 and 2x + y − z + 5 = 0.

Q.20 Find the distance between the point P(6, 5, 9) and the plane determined by the points A(3, −1, 2) B(5, 2, 4) and C(−1, −1, 6).

Q.21 Find the distance of the point (1, −2, 3) from the plane x − y + z = 5 measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Q.22 Find the coordinates of the image of the point (1, 3, 4) in the plane 2x − y + z + 3 = 0.

Q.23 From the point P(1, 2, 4), a perpendicular is drawn on the plane 2x + y − 2z + 3 = 0. Find its equation and the length. Also find the coordinates of the foot of the perpendicular.

Q.24 Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$

Q.25 Find the equation of the plane passing through the point (−1, −1, 2) and perpendicular to each of the planes 2x + 3y − 3z = 2 and 5x − 4y + z = 6

Q.26 Show that the lines $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar.

Also find the equation of the plane containing the lines

Q.27 Find the equation of the plane determined by the points A(3, −1, 2), B(5, 2, 4) and C(−1, −1, 6). Also find the distance of the point P(6, 5, 9) from the plane.

Q.28 Find the equation of the plane passing through the point (−1, 3, 2) and perpendicular to each of the plane x + 2y + 3z = 5 and 3x + 3y + z = 0.

Q.29 Find the point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point P(1, 3, 3).

Q.30 Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point P(3, 2, 1) from the plane 2x − y + z + 1 = 0. Find also, the image of the point in the plane.

Q.31 Find the shortest distance between the following lines whose vector equations are :

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and } \vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

Q.32 Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 1$ and $\vec{r}.(2\hat{i}+3\hat{j}-\hat{k}) + 4 = 0$ and parallel to x-axis.

Q.33. Find the coodinates of the point where the line through the points A(3, 4, 1) and B(5, 1), 6) crosses the XY-plane.

Q.34 If the lines $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular, find the value of k and hence find the equation of plane containing these lines.

Q.34 Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)$$

are intersecting . Hence find their point of intersection.

OR

Find the vector equation of the plane through the points (2, 1, −1) and (−1, 3, 4) and perpendicular to the plane x - 2y + 4z = 10.

Q.35 Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.\left(\hat{i} + 3\hat{j}\right) - 6 = 0$ and $\vec{r}.\left(3\hat{i} - \hat{j} - 4\hat{k}\right) = 0$, whose perpendicular distance from origin is unity.

OR

Find the vector equation of the line passing through the point (1, 2, 3) and parallewl to the planes $\vec{r}.\left(\hat{i} - \hat{j} - 2\hat{k}\right) = 5$ and $\vec{r}.\left(3\hat{i} + \hat{j} + \hat{k}\right) = 6$.

Answers

EXERCISE – 1 (UNSOLVED PROBLEMS)

1. $-\frac{4}{5}$ **2.** (7, −7, 0) **3.** (−2, 1, −1) **4.** $\left(\frac{11}{17}, \frac{-26}{17}, \frac{28}{17}\right)$; $\left(\frac{22}{17}, \frac{-52}{17}, \frac{56}{17}\right)$ **5.** $\frac{3\sqrt{70}}{28}$ units

8. $\frac{1}{\sqrt{637}}$ **10.** $\frac{10}{\sqrt{21}}$ units **11.** (3, 4, 5) **12.** $\frac{x-2}{4} = \frac{y+1}{-5} = \frac{z+3}{2}$ **13.** $y - 2 = \frac{z+3}{-2}$; x = 1

14. 2x + y − z − 6 =0 0 **15.** $\sqrt{\frac{253}{17}}$ units **16.** $\vec{r}.\left(2\hat{i} - \hat{j} + \hat{k}\right) = 30;\ 2x - y + z - 30 = 0$ **17.** 1

18. $\left(\frac{15}{13}, \frac{19}{13}, \frac{15}{13}\right)$ **19.** (8,-1,4) **20.** x = 1 - y = z **21.** (1,5,0) **22.** $7x + 4y - z - 16 = 0$

23. $x + 3y + 5z = 0$ **24.** $x - 2y + z - 5 = 0$ **25.** $13x + 10y - 19z + 9 = 0$ **26.** $11x + y + 7z - 29 = 0$

27. $10x + 13y + 2z - 1 = 0$ **28.** $3x + 9y - 2z - 14 = 0$ **29.** $5x + 2y - 3z - 17 = 0$

30. $\vec{r}.\left(5\hat{i} - 2\hat{j} - 3\hat{k}\right) = 7$

EXERCISE – 2 (BOARD PROBLEMS)

1. $\vec{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + t(6\hat{i} - 3\hat{j} + 5\hat{k})$; $\left(\frac{76}{35}, \frac{-108}{35}, \frac{-34}{7}\right)$ **2.** $\vec{r}.(15\hat{i} - 47\hat{j} + 28\hat{k}) - 7 = 0$ **3.** $\frac{10}{\sqrt{27}}$ units **4.** 7 units

5. $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$; $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$ **7.** $\frac{8}{\sqrt{29}}$ units **8.** 17x + 2y − 7z = 12 **9.** (2, 3, −1)

10. 17x + 2y − 7z = 12 **11.** $\sqrt{11}$ units ; (5, 2, 6) **12.** (−5, 6, 1) ; No **13.** (−1, −1, −1) **14.** $\frac{3}{\sqrt{19}}$ units

15. 9x − 8y + 7z − 21 = 0; $\frac{22}{\sqrt{194}}$ **16.** 3x + 14y + 6z + 49 = 0; $\vec{r}.(3\hat{i} + 14\hat{j} - 6\hat{k}) + 49 = 0$

19. 51x + 15y − 50z + 173 = 0 **20.** $\frac{6}{\sqrt{34}}$ units **21.** 1 unit **22.** (−3,5,2) **23.** 9x + 17y + 23z = 20

24. $\frac{x-4}{1} = \frac{y-6}{1} = \frac{z-2}{2}$ **25.** 2x − 13y + 3z = 0 or $\vec{r}.(2\hat{i} - 13\hat{j} + 3\hat{k}) = 0$ **27.** 3x − 4y + 3z − 19 = 0; $\frac{6}{\sqrt{34}}$ units

28. 7x − 8y + 3z + 25 = 0 **29.** (−2, −1, 3) or (4, 3, 7) **30.** (−1, 4, −1) **5.** x − 2y + z = 0

31. $\frac{8}{\sqrt{29}}$ **32.** y − 3z + 6 = 0 **33.** (13/5, 23/5, 0) **34.** 22x − 19 y − 5z + 31 = 0