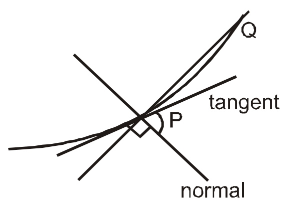
APPLICATIONS OF DERIVATIVES

TANGENTS & NORMALS

Tangent and Normal :

Let y = f(x) be function with graph as shown in figure. Consider secant PQ. If Q tends to P along the curve passing through the points Q_1, Q_2, ...I.e. $Q \to P$, secant PQ will become tangent at P. A line through P perpendicular to tangent is called normal at P.



Figure

Geometrical Meaning of $\frac{dy}{dx}$:

As $Q \to P$, $h \to 0$ and slope of chord PQ tends to slope of tangent at P (see figure).

Slope of chord PQ = $\frac{f(x+h)-f(x)}{h}$

$\lim_{Q\to P}$ slope of chord PQ = $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow$ slope of tangent at P = $f'(x) = \frac{dy}{dx}$

Equation of tangent and normal :

$\left.\frac{dy}{dx}\right]_{(x_1, y_1)} = f'(x_1)$ denotes the slope of tangent at point (x_1, y_1) on the curve y = f(x). Hence

the equation of tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

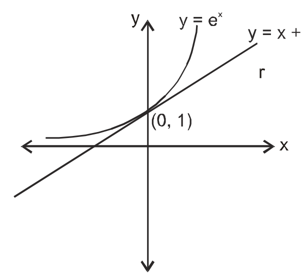
Also, since normal is a line perpendicular to tangent at (x_1, y_1) so its equation is given by

$(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

If $f'(x_1) = 0$, then tangent is the line $y = y_1$ and normal is the line $x = x_1$.

If $\lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{h} = \infty$ or $-\infty$, then $x = x_1$ is tangent **(VERTICAL TANGENT)** and $y = y_1$ is normal.

Ex. 1 Find equation of tangent to $y = e^x$ at $x = 0$. Hence draw graph



Sol. At $x = 0 \Rightarrow y = e^0 = 1$

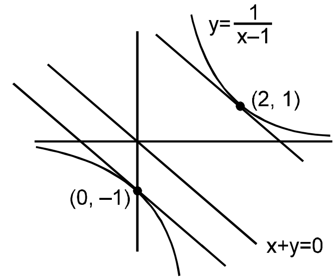
$\frac{dy}{dx} = e^x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 1$

Hence equation of tangent is

$1(x-0) = (y-1) \Rightarrow y = x + 1$

Ex. 2 Find the equation of all straight lines which are tangent to curve $y = \frac{1}{x-1}$ and which are parallel to the line $x + y = 0$.

Sol. Suppose the tangent is at (x_1, y_1) and it has slope – 1.



$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -1.$

$\Rightarrow -\frac{1}{(x_1-1)^2} = -1.$

$\Rightarrow x_1 = 0$ or 2

$\Rightarrow y_1 = -1$ or 1

Hence tangent at (0, – 1) and (2, 1) are the required lines (see figure) with equations

$-1(x-0) = (y+1)$ and $-1(x-2) = (y-1)$

$\Rightarrow x + y + 1 = 0$ and $y + x = 3$

Ex.3 Find equation of normal to the curve $y = |x^2 - |x||$ at $x = -2$.

Sol. In the neighborhood of $x = -2$, $y = x^2 + x$.

Hence the point of contact is (– 2, 2)

$\frac{dy}{dx} = 2x + 1 \Rightarrow \left.\frac{dy}{dx}\right|_{x=-2} = -3.$

So the slope of normal at (– 2, 2) is .

Hence equation of normal is

$\frac{1}{3}$ (x + 2) = y – 2

$\Rightarrow \quad 3y = x + 8$

Ex.4 Find the equation of tangent and normal to the curve $x = \frac{2at^2}{1+t^2}$, $y = \frac{2at^3}{1+t^2}$ at the point $t = \frac{1}{2}$.

Sol. Given that

$x = \frac{2at^2}{1+t^2} \qquad y = \frac{2at^3}{1+t^2}$

at $t = \frac{1}{2}$, $\qquad x = \frac{2a}{5}$, $\qquad y = \frac{a}{5}$

also $\frac{dx}{dt} = \frac{4at}{(1+t^2)^2}$ and $\frac{dy}{dt} = \frac{2at^2(3+t^2)}{(1+t^2)^2}$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2}\, t\,(3 + t^2)$

when $t = \frac{1}{2}$, $\frac{dy}{dx} = \frac{1}{2}\frac{1}{2}\left(3+\frac{1}{4}\right) = \frac{13}{16}$

$\therefore$ The equation of the tangent when $t = \frac{1}{2}$ is

$y - \frac{a}{5} = \left(\frac{13}{16}\right)\left(x - \frac{2a}{5}\right)$

$\Rightarrow 13x - 16y = 2a$

and the equation of the normal is $\left(y - \frac{a}{5}\right)\left(\frac{13}{16}\right) + x - \frac{2a}{5} = 0$

$\Rightarrow 16x + 13y = 9a$

Ex. 5 Find value of 'c' such that line joining (0, 4) and (5, –1) become tangent to curve $y = \frac{c}{x+1}$.

Sol. Equation of line joining A & B is x + y = 4

Solving this line and curve we get

$4 - x = \frac{c}{x+1} \quad \Rightarrow \quad x^2 - 3x + (c - 4) = 0$ ......(i)

For tangency, roots of this equation must be equal.

Hence discriminant of quadratic equation = 0

$\Rightarrow \quad 9 = 4(c - 4) \quad \Rightarrow \quad \frac{9}{4} = c - 4$

$c = \frac{9}{4} + 4 \quad \Rightarrow \quad c = \frac{25}{4}$

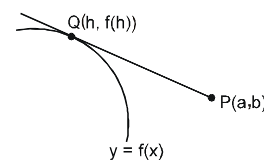
$x^2 - 3x + \frac{9}{4} = 0 \quad \Rightarrow \quad x^2 - 2.\ \frac{3}{2}x + \frac{9}{4} - \frac{9}{4} + \frac{9}{4} = 0$

$\left(x - \frac{3}{2}\right)^2 = 0 \quad \Rightarrow \quad x = \frac{3}{2}$

hence point of contact becomes $\left(\frac{3}{2}, \frac{5}{2}\right)$

Tangent and Normal from an external point :

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.



Figure

$f'(h) = \frac{f(h) - b}{h - a}$

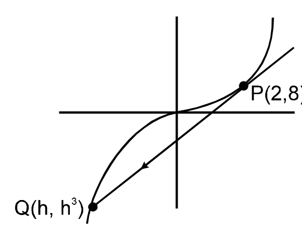
And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

Ex.6 Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Sol. Equation of tangent at (2, 8) is $y = 12x - 16$

Solving this with $y = x^3$

$x^3 - 12x + 16 = 0$



This cubic will give all points of intersection of line and curve $y = x^3$ i.e., point P and Q. (see figure)

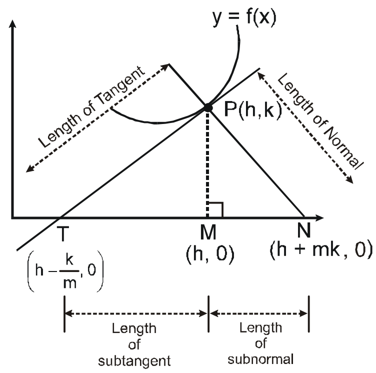
But, since line is tangent at P so x = 2 will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be x = h. Using theory of equations :

sum of roots $\Rightarrow \quad 2 + 2 + h = 0 \quad \Rightarrow \quad h = -4$

Hence coordinates of Q are (– 4, – 64)

Lengths of tangent, normal, subtangent and subnormal :

Let P (h, k) be any point on curve y = f(x). Let tangent drawn at point P meets x-axis at T & normal at point P meets x-axis at N. Then the length PT is called the length of tangent and PN is called length of normal. (as shown in figure)



Projection of segment PT on x-axis, TM, is called the subtangent and similarly projection of line segment PN on x axis, MN is called subnormal. Let $m = \left.\frac{dy}{dx}\right]_{(h,k)}$ = slope of tangent.

Hence equation of tangent is $m(x - h) = (y - k)$.

Putting y = 0, we get x - intercept of tangent is $x = h - \frac{k}{m}$

Similarly, the x-intercept of normal is x = h + km

Now, length PT, PN,TM, MN can be easily evaluated using distance formula

(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1+m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) MN = |km| = Length of subnormal.

Ex. 7 Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point x = 0.

Sol. Here, $m = \left.\frac{dy}{dx}\right|_{x=0}$

$\frac{dy}{dx} = 3x^2 + 6x + 4 \quad \Rightarrow \quad m = 4$

and, $k = y(0) \quad \Rightarrow \quad k = -1$

$\lambda = |k|\sqrt{1 + \frac{1}{m^2}} \quad \Rightarrow \quad \lambda = |(-1)|\sqrt{1+\frac{1}{16}} = \frac{\sqrt{17}}{4}$

Ex.8 Determine 'p' such that the lenght of the sub-tangent and sub-normal is equal for the curve $y = e^{px} + px$ at the point (0, 1)

Sol. $\frac{dy}{dx} = pe^{px} + p$ at point $(0, 1) = 2p$

subnormal $= \left|y\frac{dy}{dx}\right|$

subtangent $= \left|y\frac{dx}{dy}\right|$

$\frac{dy}{dx} = \pm 1 \quad \Rightarrow \quad 2p = \pm 1 \quad \Rightarrow \quad p = \pm\frac{1}{2}$

Ex.9 For the curve $y = a\,\lambda n\,(x^2 - a^2)$ show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.

Sol. Let point of tangency be (x_1, y_1)

$m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{2ax_1}{x_1^2 - a^2}$

Length of tangent + subtangent $= |y_1|\sqrt{1 + \frac{1}{m^2}} + \left|\frac{y_1}{m}\right|$

$= |y_1|\sqrt{1 + \frac{(x_1^2 - a^2)^2}{4a^2x_1^2}} + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right|$

$= |y_1|\frac{\sqrt{x_1^4 + a^4 + 2a^2x_1^2}}{2|ax_1|} + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right|$

$= \left|\frac{y_1(x_1^2 + a^2)}{2ax_1}\right| + \left|\frac{y_1(x_1^2 - a^2)}{2ax_1}\right|$

$= \frac{|y_1|(2x_1^2)}{2|ax_1|} = \left|\frac{x_1\ y_1}{a}\right|$