APPLICATIONS OF DERIVATIVES

MAXIMA & MINIMA

Global Maximum :

A function f(x) is said to have global maximum on a set E if there exists at least one $c \in E$ such that $f(x) \leq f(c)$ for all $x \in E$.

We say global maximum occurs at $x = c$ and global maximum (or global maximum value) is f(c).

Local Maxima :

A function f(x) is said to have a local maximum at $x = c$ if f(c) is the greatest value of the function in a small neighborhood $(c - h, c + h)$, $h > 0$ of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \leq f(c)$.

Note : If $x = c$ is a boundary point then consider $(c - h, c)$ or $(c, c + h)$ $(h > 0)$ appropriately.

Global Minimum :

A function f(x) is said to have a global minimum on a set E if there exists at least one $c \in E$ such that $f(x) \geq f(c)$ for all $x \in E$.

Local Minima :

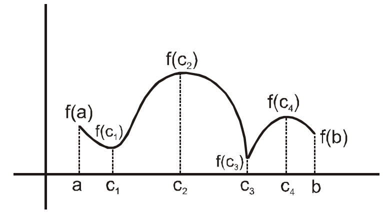
A function f(x) is said to have a local minimum at $x = c$ if f(c) is the least value of the function in a small neighbourhood $(c - h, c + h)$, $h > 0$ of c.

i.e. for all $x \in (c - h, c + h)$, $x \neq c$, we have $f(x) \geq f(c)$.

Extrema :

A maxima or a minima is called an extrema.

Explanation : Consider graph of $y = f(x)$, $x \in [a, b]$



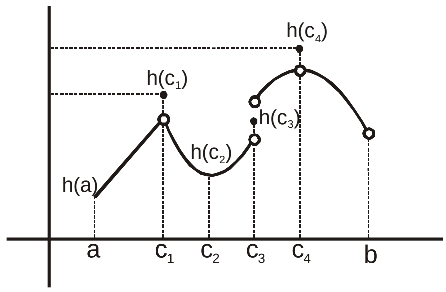
$x = a$, $x = c_2$, $x = c_4$ are points of local maxima, with maximum values $f(a)$, $f(c_2)$, $f(c_4)$ respectively.

$x = c_1$, $x = c_3$, $x = b$ are points of local minima, with minimum values $f(c_1)$, $f(c_3)$, $f(b)$ respectively

$x = c_2$ is a point of global maximum

$x = c_3$ is a point of global minimum

Consider the graph of $y = h(x)$, $x \in [a, b)$



$x = c_1$, $x = c_4$ are points of local maxima, with maximum values $h(c_1)$, $h(c_4)$ respectively.

$x = a$, $x = c_2$ are points of local minima, with minimum values $h(a)$, $h(c_2)$ respectively.

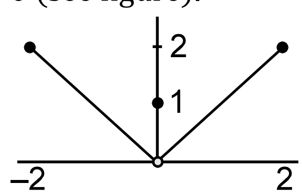
$x = c_3$ is neither a point of maxima nor a minima.

Global maximum is $h(c_4)$

Global minimum is $h(a)$

Ex.1 Let $f(x) = \begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behaviour of $f(x)$ at $x = 0$.

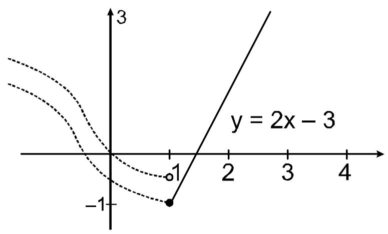
Sol. $f(x)$ has local maxima at $x = 0$ (see figure).



Ex.2 Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$

Find all possible values of b such that $f(x)$ has the smallest value at $x = 1$.

Sol. Such problems can easily be solved by graphical approach (as in figure).



Hence the limiting value of f(x) from left of x = 1 should be either greater or equal to the value of function at x = 1.

$\lim_{x \to 1^-} f(x) \geq f(1)$

$\Rightarrow \quad -1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \geq -1$

$\Rightarrow \quad \frac{(b^2 + 1)(b - 1)}{(b + 1)\ (b + 2)} \geq 0$

$\Rightarrow \quad b \in (-2, -1) \cup [1, +\infty)$

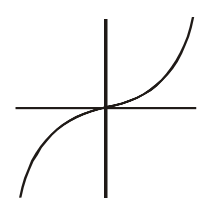
Maxima, Minima for differentiable functions :

Mere definition of maxima, minima becomes tedious in solving problems. We use derivative as a tool to overcome this difficulty.

(i) A necessary condition for an extrema : Let f(x) be differentiable at x = c.

Theorem : A necessary condition for f(c) to be an extremum of f(x) is that $f'(c) = 0$.

i.e. f(c) is extremum $\Rightarrow$ $f'(c) = 0$



Note : $f'(c) = 0$ is only a necessary condition but not sufficient

i.e. $f'(c) = 0 \Rightarrow$ f(c) is extremum.

Consider $f(x) = x^3$

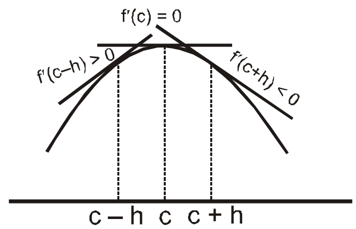
$f'(0) = 0$

but f(0) is not an extremum (see figure).

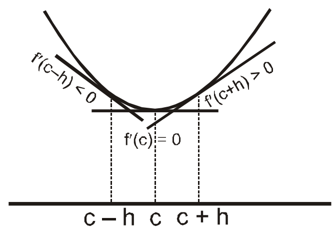
(ii) Sufficient condition for an extrema : Let f(x) be a differentiable function.

Theorem : A sufficient condition for f(c) to be an extremum of f(x) is that $f'(x)$ changes sign as x passes through c.

i.e. f(c) is an extrema (see figure) $\Leftrightarrow$ $f'(x)$ changes sign as x passes through c.



x = c is a point of maxima. f′(x) changes sign from positive to negative.



x = c is a point of local minima (see figure), f′(x) changes sign from negative to positive.

Stationary points :

The points on graph of function f(x) where $f'(x) = 0$ are called stationary points.

Rate of change of f(x) is zero at a stationary point.

Ex.3 Find stationary points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$.

Sol. $f'(x) = 12x^2 - 12x - 24$

$f'(x) = 0 \quad \Rightarrow \quad x = -1, 2$

$f(-1) = 23, f(2) = -31$

$(-1, 23), (2, -31)$ are stationary points

Ex.4 If $f(x) = x^3 + ax^2 + bx + c$ has extreme values at $x = -1$ and $x = 3$. Find a, b, c.

Sol. Extreme values basically mean maximum or minimum values, since f(x) is differentiable function so

$f'(-1) = 0 = f'(3)$

$f'(x) = 3x^2 + 2ax + b$

$f'(3) = 27 + 6a + b = 0$

$f'(-1) = 3 - 2a + b = 0$

$\Rightarrow a = -3, b = -9, c \in R$

First Derivative Test :

Let f(x) be continuous and differentiable function.

Step I. Find $f'(x)$

Step II. Solve $f'(x) = 0$, let $x = c$ be a solution. (i.e. Find stationary points)

Step III. Observe change of sign

(i) If $f'(x)$ changes sign from negative to positive as x crosses c from left to right then $x = c$ is a point of local minima

(ii) If $f'(x)$ changes sign from positive to negative as x crosses c from left to right then $x = c$ is a point of local maxima.

(iii) If $f'(x)$ does not changes sign as x crosses c then $x = c$ is neither a point of maxima nor minima.

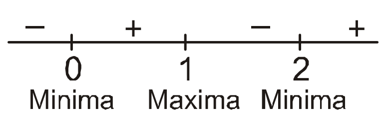
Ex.5 Find the points of maxima or minima of $f(x) = x^2 (x - 2)^2$.

Sol. $f(x) = x^2 (x - 2)^2$

$f'(x) = 4x (x - 1) (x - 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 2$

examining the sign change of $f'(x)$



Hence $x = 1$ is point of maxima, $x = 0, 2$ are points of minima.

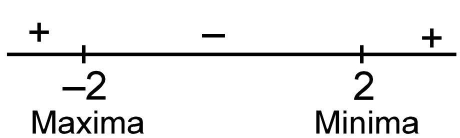
Note : In case of continuous functions points of maxima and minima are alternate.

Ex.6 Find the points of maxima, minima of $f(x) = x^3 - 12x$. Also draw the graph of this functions.

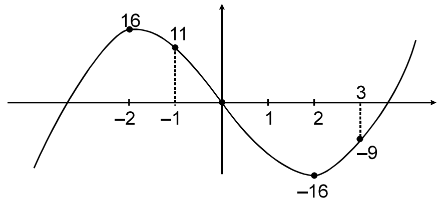
Sol. $f(x) = x^3 - 12x$

$f'(x) = 3(x^2 - 4) = 3(x - 2) (x + 2)$

$f'(x) = 0 \quad \Rightarrow \quad x = \pm 2$



For tracing the graph let us find maximum and minimum values of f(x).



x	f(x)
2	–16
–2	+16

Ex.7 Show that $f(x) = (x^3 - 6x^2 + 12x - 8)$ does not have any point of local maxima or minima. Hence draw graph

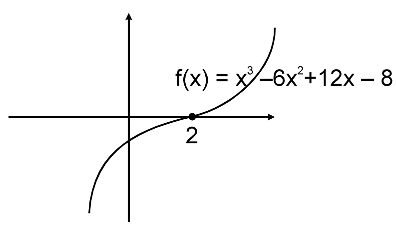
Sol. $f(x) = x^3 - 6x^2 + 12x - 8$

$f'(x) = 3(x^2 - 4x + 4)$

$f'(x) = 3(x - 2)^2$

$f'(x) = 0 \quad \Rightarrow \quad x = 2$

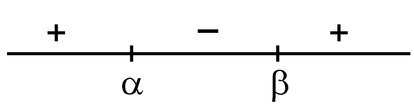
but clearly $f'(x)$ does not change sign about $x = 2$. $f'(2^+) > 0$ and $f'(2^-) > 0$. So f(x) has no point of maxima or minima. In fact f(x) is a monotonically increasing function for $x \in R$.



Ex.8 Let $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If f(x) has positive point of maxima, then find possible values of 'a'.

Sol. $f'(x) = 3[x^2 + 2(a - 7)x + (a^2 - 9)]$

Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.



Maxima occurs at smaller root α which has to be positive. This basically implies that both roots of $f'(x) = 0$ must be positive and distinct.

(i) $D > 0 \quad \Rightarrow \quad a < \frac{29}{7}$

(ii) $-\frac{b}{2a} > 0 \quad \Rightarrow \quad a < 7$

(iii) $f'(0) > 0 \quad \Rightarrow \quad a \in (-\infty, -3) \cup (3, \infty)$

from (i), (ii) and (iii) $\quad \Rightarrow \quad a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

Application of Maxima, Minima :

For a given problem, an objective function can be constructed in terms of one parameter and then extremum value can be evaluated by equating the differential to zero. As discussed in n^{th} derivative test maxima/minima can be identified.

(i) Useful Formulae of Mensuration to Remember :

(a) Volume of a cuboid $= \lambda bh$.

(b) Surface area of cuboid $= 2(\lambda b + bh + h\lambda)$.

(c) Volume of cube $= a^3$

(d) Surface area of cube $= 6a^2$

(e) Volume of a cone $= \frac{1}{3} \pi r^2 h$.

(f) Curved surface area of cone $= \pi r\lambda$ (λ = slant height)

(g) Curved surface area of a cylinder $= 2\pi rh$.

(h) Total surface area of a cylinder $= 2\pi rh + 2\pi r^2$.

(i) Volume of a sphere $= \pi r^3$.

(j) Surface area of a sphere $= \frac{4}{3} 4\pi r^2$.

(k) Area of a circular sector $= \frac{1}{2} r^2 \theta$, when θ is in radians.

(l) Volume of a prism = (area of the base) × (height).

(m) Lateral surface area of a prism = (perimeter of the base) × (height).

(n) Total surface area of a prism = (lateral surface area) + 2 (area of the base)

(Note that lateral surfaces of a prism are all rectangle).

(o) Volume of a pyramid $= \frac{1}{3}$ (area of the base) × (height).

(p) Curved surface area of a pyramid $= \frac{1}{2}$ (perimeter of the base) × (slant height).

(Note that slant surfaces of a pyramid are triangles).

Ex.9 If the equation $x^3 + px + q = 0$ has three real roots, then show that $4p^3 + 27q^2 < 0$.

Sol. $f(x) = x^3 + px + q$, $f'(x) = 3x^2 + p$

$\therefore$ f(x) must have one maximum > 0 and one minimum < 0. $f'(x) = 0$

$\Rightarrow x = \pm\sqrt{\frac{-p}{3}}, \quad p < 0$

f is maximum at $x = -\sqrt{\frac{-p}{3}}$ and minimum at $x = \sqrt{\frac{-p}{3}}$

$f\left(-\sqrt{\frac{-p}{3}}\right) \ f\left(\sqrt{\frac{-p}{3}}\right) < 0 \quad \Rightarrow \quad \left(q - \frac{2p}{3}\sqrt{\frac{-p}{3}}\right)\left(q + \frac{2p}{3}\sqrt{\frac{-p}{3}}\right) < 0$

$q^2 + \frac{4p^3}{27} < 0, \ 4p^3 + 27q^2 < 0.$

Ex.10 Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Sol. $x + y = 60$

$\Rightarrow x = 60 - y$

$\Rightarrow xy^3 = (60 - y)\, y^3$

Let $f(y) = (60 - y)\, y^3 \quad ; \quad y \in (0, 60)$

for maximizing f(y) let us find critical points

$f'(y) = 3y^2 (60 - y) - y^3 = 0$

$f'(y) = y^2 (180 - 4y) = 0$

$\Rightarrow y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at $y = 45$.

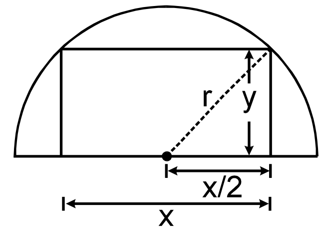
So $x = 15$ and $y = 45$.

Ex.11 Rectangles are inscribed inside a semicircle of radius r. Find the rectangle with maximum area.

Sol. Let sides of rectangle be x and y (as shown in figure).

$\Rightarrow \quad A = xy.$

Here x and y are not independent variables and are related by Pythogorus theorem with r.



$\frac{x^2}{4} + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - \frac{x^2}{4}}$

$\Rightarrow \quad A(x) = x\sqrt{r^2 - \frac{x^2}{4}} \quad \Rightarrow \quad A(x) = \sqrt{x^2r^2 - \frac{x^4}{4}}$

Let $f(x) = r^2x^2 - \frac{x^4}{4}$; $x \in (0, r)$

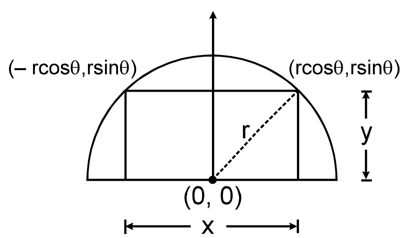
A(x) is maximum when f(x) is maximum

Hence $f'(x) = x(2r^2 - x^2) = 0 \quad \Rightarrow \quad x = r\sqrt{2}$

also $f'(r\sqrt{2^+}) < 0$ and $f'(r\sqrt{2^-}) > 0$

confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \frac{r}{\sqrt{2}}$.

Aliter : Let us choose coordinate system with origin as centre of circle (as shown in figure).

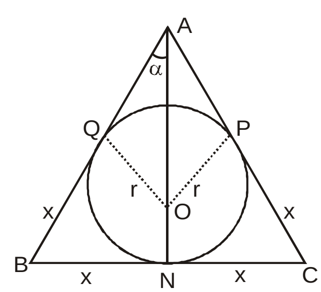


$A = xy$

$\Rightarrow A = 2\,(r\cos\theta)\,(r\sin\theta) \quad \Rightarrow \quad A = r^2 \sin 2\theta, \ \theta \in \left(0, \frac{\pi}{2}\right)$

Clearly A is maximum when $\theta = \frac{\pi}{4} \quad \Rightarrow \quad x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Ex.12 Show that the least perimeter of an isosceles triangle circumscribed about a circle of radius 'r' is $6\sqrt{3}\ r$.



Sol. $AQ = r\cot\alpha = AP$

$AO = r\,\text{cosec}\alpha$

$\frac{x}{AO + ON} = \tan\alpha$

$x = (r\,\text{cosec}\alpha + r)\tan\alpha$

$x = r(\sec\alpha + \tan\alpha)$

Perimeter $= p = 4x + 2AQ$

$p = 4r(\sec\alpha + \tan\alpha) + 2r\cot\alpha$

$p = r(4\sec\alpha + 4\tan\alpha + 2\cot\alpha)$

$\frac{dp}{d\alpha} = r[4\sec\alpha \tan\alpha + 4\sec^2\alpha - 2\text{cosec}^2\alpha]$

for max or min $\frac{dp}{d\alpha} = 0 \quad \Rightarrow \quad 2\sin^3\alpha + 3\sin^2\alpha - 1 = 0$

$\Rightarrow (\sin\alpha + 1)(2\sin^2\alpha + \sin\alpha - 1) = 0$

$(\sin\alpha + 1)^2 (2\sin\alpha - 1) = 0$

$\Rightarrow \quad \sin\alpha = 1/2$

$\Rightarrow \quad \alpha = 30° = \pi/6$

pleast $= r = \left[\frac{4.2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + 2\sqrt{3}\right] r = \left[\frac{8+4+6}{\sqrt{3}}\right] r \; \frac{(6\sqrt{3}\sqrt{3})}{\sqrt{3}} = 6\sqrt{3} \quad r$

Ex.13 If a right circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Sol. Let x be the radius of cylinder and y be its height $v = \pi x^2 y$

x, y can be related by using similar triangles

(as shown in figure).

$\frac{y}{r-x} = \frac{h}{r} \quad \Rightarrow \quad y = \frac{h}{r}(r - x)$

$\Rightarrow v(x) = \pi x^2 \frac{h}{r}(r - x) \qquad x \in (0, r)$

$\Rightarrow v(x) = \frac{\pi h}{r}(rx^2 - x^3)$

$v'(x) = \frac{\pi h}{r} x (2r - 3x)$

$v' = \left(\frac{2r}{3}\right) = 0$ and $v''\left(\frac{2r}{3}\right) < 0$

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Note : Following formulae of volume, surface area of important solids are very useful in problems of maxima & minima.

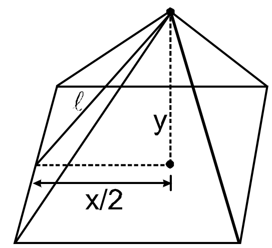
Ex.14 Among all regular square pyramids of volume $36\sqrt{2}$ cm^3. Find dimensions of the pyramid having least lateral surface area.

Sol. Let the length of a side of base be x cm and y be the perpendicular height of the pyramid (see figure).

$V = \frac{1}{3} \times$ area of base x height $\Rightarrow$ $V = \frac{1}{3} x^2y = 36\sqrt{2} \Rightarrow y = \frac{108\sqrt{2}}{x^2}$

and $S = \frac{1}{2} \times$ perimeter of base x slant height $= \frac{1}{2}(4x).\lambda$

but $\lambda = \sqrt{\frac{x^2}{4} + y^2}$



$\Rightarrow$ $S = 2x\sqrt{\frac{x^2}{4} + y^2} = \sqrt{x^4 + 4x^2y^2}$

$\Rightarrow$ $S = \sqrt{x^4 + 4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2}$ $S(x) = \sqrt{x^4 + \frac{8.(108)^2}{x^2}}$

Let $f(x) = x^4 + \frac{8.(108)^2}{x^2}$ for minimizing f(x)

$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0$

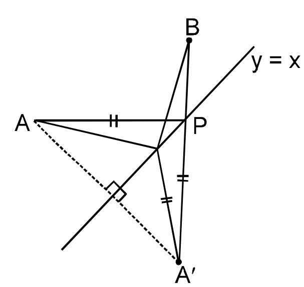
$\Rightarrow$ $f'(x) = 4\frac{(x^6 - 6^6)}{x^3} = 0$

$\Rightarrow$ x = 6, which a point of minima. Hence x = 6 cm and $y = 3\sqrt{2}$.

Ex.15 Let A(1, 2) and B(– 2, – 4) be two fixed points. A variable point P is chosen on the straight line y = x such that perimeter of ΔPAB is minimum. Find coordinates of P.

Sol. Since distance AB is fixed so for minimizing the perimeter of ΔPAB, we basically have to minimize (PA + PB)

Let A′ be the mirror image of A in the line y = x (see figure).



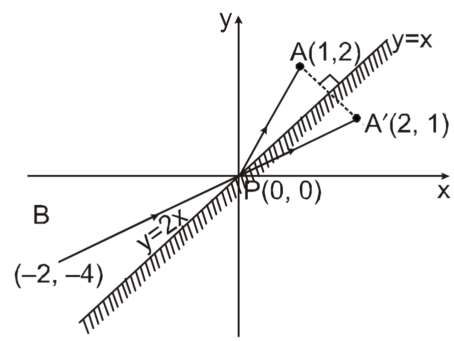
F(P) = PA + PB

F(P) = PA′ + PB

But for ΔPA′B

PA′ + PB ≥ A′B and equality hold when P, A′ and B becomes collinear. Thus for minimum path length point P is that special point for which PA and PB become incident and reflected rays with respect to the mirror y = x.

Equation of line joining A′ and B is y = 2x intersection of this line with y = x is the point P. Hence P ≡ (0, 0).



Note : Above concept is very useful because such problems become very lengthy by making perimeter as a function of position of P and then minimizing it.