APPLICATIONS OF DERIVATIVES

INTRODUCTION, RATE OF CHANGE OF QUANTITIES

Derivative as rate of change :

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.
An economist may want to study how the investment changes with respect to variations in interest rates. A physician may want to know, how small changes in dosage can affect the body's response to a drug.

Rate of Change of Quantities

Let y = f(x) be a function of x and x is a function (of time), then y is also a function of t.

Now $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{rat\ of\ \ change\ of\ \ y}{rat\ of\ \ change\ of\ \ x}$$

$\frac{dy}{dx}$ is equal to the ratio of the rate of change of y and rate of change of x. So $\frac{dy}{dx}$ represents instantaneous rate of change in y with respect to x.

VELOCITY AND ACCELERATION

$Vrlocity = \frac{ds}{dt} = \lim_{\delta t \to 0} \frac{\delta s}{\delta t}$ and $acceleration = \frac{d^2 s}{dt^2}$

where s is distance covered in time t and s + δs is the distance covered in time t + δt

Approximate change in the value of the function y = f(x) $\delta y = \frac{dy}{dx} \delta x$

where δy = approximate change in y.

Approximate value of function y = f(x) at x = a, f (a +h) =f(a)+hf'(a) where h $\to$ 0

Ex .1 How fast the area of a circle increases when its radius is 5cm;

(i) with respect to radius

(ii) with respect to diameter

Sol. (i) $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$

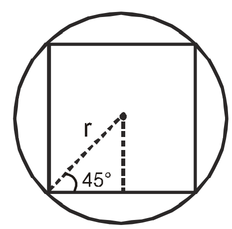
$\therefore \left.\frac{dA}{dr}\right]_{r=5} = 10\pi\ cm^2/cm.$

(ii) $A = \frac{\pi}{4} D^2$, $\frac{dA}{dD} = \frac{\pi}{2} D$

$\therefore \left.\frac{dA}{dD}\right]_{D=10} = \frac{\pi}{2} . 10 = 5\pi\ cm^2/cm.$

Ex.2 If area of circle increases at a rate of $2cm^2/sec$, then find the rate at which area of the inscribed square increases.

Sol. Area of circle, $A_1 = \pi r^2$. Area of square, $A_2 = 2r^2$ (see figure)



$\frac{dA_1}{dt} = 2\pi r \frac{dr}{dt}$, $\Rightarrow$ $\frac{dA_2}{dt} = 4r . \frac{dr}{dt}$

$\because$ $2 = 2\pi r . \frac{dr}{dt}$ $\Rightarrow$ $r \frac{dr}{dt} = \frac{1}{\pi}$

$\therefore$ $\frac{dA_2}{dt} = 4 . \frac{1}{\pi} = \frac{4}{\pi}$ cm^2/sec

$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm^2/sec.

Ex.3 The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Sol. Let at some time t, the length of edge is x cm.

$v = x^3$

$\Rightarrow$ $\frac{dv}{dt} = 3x^2 \frac{dx}{dt}$ (but $\frac{dv}{dt} = 7$)

$\Rightarrow$ $\frac{dx}{dt} = \frac{7}{3x^2}$ cm/sec.

Now $S = 6x^2$

$\frac{dS}{dt} = 12x \frac{dx}{dt}$

$\Rightarrow \quad \frac{dS}{dt} = 12x. \frac{7}{3x^2} = \frac{28}{x}$

when x = 4 cm, $\frac{dS}{dt} = 7$ cm^2/sec.

Ex.4 Sand is pouring from pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Sol. $V = \frac{1}{3} \pi r^2 h$ but $h = \frac{r}{6}$

$\Rightarrow V = \frac{1}{3} \pi (6h)^2 h$

$\Rightarrow V = 12\pi h^3$

$\Rightarrow \frac{dV}{dt} = 36\pi h^2. \frac{dh}{dt}$

when, $\frac{dV}{dt} = 12$ cm^3/s and h = 4 cm

$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi}$ cm/sec.