APPLICATIONS OF DERIVATIVES

INCREASING & DECREASING FUNCTIONS

Monotonicity of a function:

Let f be a real valued function having domain D(DR) and S be a subset of D. f is said to be monotonically increasing (non- decreasing) (increasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$. f is said to be monotonically decreasing (non- increasing) (decreasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Notes :

(i) f is strictly increasing $\Rightarrow$ f is monotonically increasing (non- decreasing). But converse need not be true.

(ii) f is strictly decreasing $\Rightarrow$ f is monotonically decreasing (non -increasing). Again, converse need not be true.

(iii) If f(x) = constant in S, then f is increasing as well as decreasing in S

(iv) A function f is said to be an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing

(v) f is said to be a monotonic function if either it is monotonically increasing or monotonically decreasing

(vi) If f is increasing in a subset of S and decreasing in another subset of S, then f is non monotonic in S.

Application of differentiation for detecting monotonicity :

Let I be an interval (open or closed or semi open and semi closed)

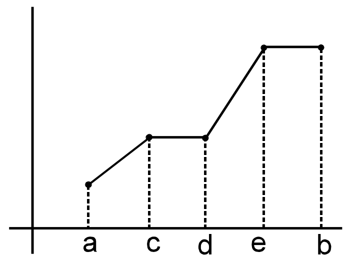
(i) If $f'(x) > 0 \ \forall \ x \in I$, then f is strictly increasing in I

(ii) If $f'(x) < 0 \ \forall \ x \in I$, then f is strictly decreasing in I

Note : Let I be an interval (or ray) which is a subset of domain of f. If $f'(x) > 0, \forall x \in I$, except for countably many points where $f'(x) = 0$, then f(x) is strictly increasing in I.

$\{f'(x) = 0$ at countably many points $\Rightarrow f'(x) = 0$ does not occur on an interval which is a subset of I $\}$

Let us consider another function whose graph is shown below for $x \in (a, b)$.



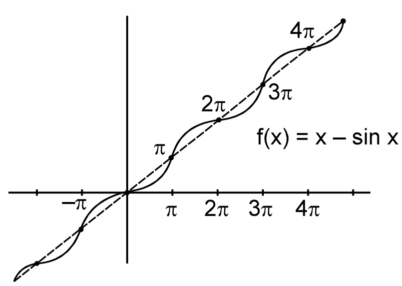
Here also $f'(x) \geq 0$ for all $x \in (a, b)$. But, note that in this case, $f'(x) = 0$ holds for all $x \in (c, d)$ and (e,b).

Thus the given function is increasing (monotonically increasing) in (a, b), but not strictly increasing.

Ex. 1 Let $f(x) = x - \sin x$. Find the intervals of monotonicity.

Sol. $f'(x) = 1 - \cos x$

Now, $f'(x) > 0$ every where, except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. But all these points are discrete (countable) and do not form an interval. Hence we can conclude that f(x) is strictly increasing in R. In fact we can also see it graphically.



Ex.2 Find the least value of k for which the function $x^2 + kx + 1$ is an increasing function in the interval $1 < x < 2$.

Sol. $f(x) = x^2 + kx + 1$

for f(x) to be increasing, $f'(x) > 0$

$\Rightarrow \frac{d}{dx}(x^2 + kx + 1) > 0$

$2x + k > 0 \qquad \Rightarrow \qquad k > -2x$

for $x \in (1, 2)$ the least value of k is -2

Ex.3 Find the intervals in which $f(x) = x^3 - 3x + 2$ is increasing.

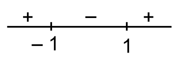
Sol. $f(x) = x^3 - 3x + 2$

$\Rightarrow \quad f'(x) = 3(x^2 - 1)$

$\Rightarrow \quad f'(x) = 3(x - 1)(x + 1)$

for M.I. $\quad f'(x) \geq 0$

$\Rightarrow \quad 3(x - 1)(x + 1) \geq 0$



$\Rightarrow \quad x \in (-\infty, -1] \cup [1, \infty)$, thus f is increasing in $(-\infty, -1]$ and also in $[1, \infty)$

Ex.4 Find the intervals of monotonicity of the following functions.

(i) $f(x) = x^2 (x - 2)^2$

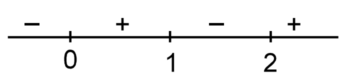
(ii) $f(x) = x \lambda n\, x$

(iii) $f(x) = \sin x + \cos x$; $x \in [0, 2\pi]$

Sol. (i) $f(x) = x^2 (x - 2)^2$

$f'(x) = 4x (x - 1)(x - 2)$

observing the sign change of $f'(x)$



Hence increasing in $[0, 1]$ and in $[2, \infty)$ and decreasing for $x \in (-\infty, 0]$ and $[1, 2]$

(ii) $f(x) = x \lambda n\, x \quad \Rightarrow \quad f'(x) = 1 + \lambda n\, x$

$f'(x) \geq 0 \quad \Rightarrow \quad \lambda n\, x \geq -1 \quad \Rightarrow \quad x \geq \frac{1}{e}$

$\Rightarrow$increasing for $x \in \left[\frac{1}{e}, \infty\right)$ and decreasing for $x \in \left(0, \frac{1}{e}\right]$.

(iii) $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

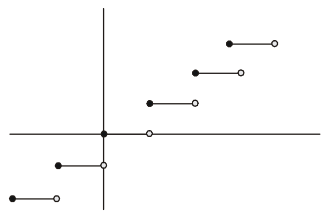
for increasing $f'(x) \geq 0 \qquad \Rightarrow \cos x \geq \sin x$

$\Rightarrow$ f is increasing in $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$ f is decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Note : If a function f(x) is increasing in (a, b) and f(x) is continuous in [a, b], then f(x) is increasing on [a, b]

Ex.5 f(x) = [x] is a step up function. Is it a strictly increasing function for $x \in R$.

Sol. No, f(x) = [x] is increasing (monotonically increasing) (non-decreasing), but not strictly increasing function as illustrated by its graph.



Ex.6 If $f(x) = \sin^4 x + \cos^4 x + bx + c$, then find possible values of b and c such that f(x) is monotonic for all $x \in R$

Sol. $f(x) = \sin^4 x + \cos^4 x + bx + c$

$f'(x) = 4 \sin^3 x \cos x - 4\cos^3 x \sin x + b = -\sin 4x + b.$

Case(i) : for M.I. $f'(x) \ge 0$ for all $x \in R$

$\Leftrightarrow$ $b \ge \sin 4x$ for all $x \in R$ $\Leftrightarrow$ $b \ge 1$

Case (ii) : for M.D. $f'(x) \le 0$ for all $x \in R$

$\Leftrightarrow$ $b \le \sin 4x$ for all $x \in R$ $\Leftrightarrow$ $b \le -1$

Hence for f(x) to be monotonic $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$.

Ex.7 Find possible values of 'a' such that $f(x) = e^{2x} - (a + 1) e^x + 2x$ is monotonically increasing for $x \in R$

Sol. $f(x) = e^{2x} - (a + 1) e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1) e^x + 2$

Now, $2e^{2x} - (a + 1) e^x + 2 \ge 0$ for all $x \in R$

$\Rightarrow$ $2\left(e^x + \frac{1}{e^x}\right) - (a + 1) \ge 0$ for all $x \in R$

$(a + 1) \le 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow \quad a + 1 \le 4 \qquad \left(\because e^x + \frac{1}{e^x} \text{ has minimum value } 2\right)$

$\Rightarrow \quad a \le 3$

Aliter (Using graph)

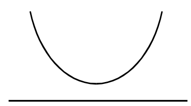
$2e^{2x} - (a + 1)\, e^x + 2 \ge 0$ for all $x \in R$

putting $e^x = t;\ \ t \in (0, \infty)$

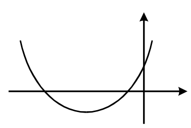
$2t^2 - (a + 1)\, t + 2 \ge 0$ for all $t \in (0, \infty)$

Case - (i) : $D \le 0$

$\Rightarrow \quad (a + 1)^2 - 4 \le 0$



$\Rightarrow \quad (a + 5)\,(a - 3) \le 0$



$\Rightarrow \quad a \in [-5, 3]$

Case - (ii) : both roots are non positive

$D \ge 0 \quad \& \quad -\frac{b}{2a} < 0 \quad \& \quad f(0) \ge 0$

$\Rightarrow \quad a \in (-\infty, -5] \cup [3, \infty) \quad \& \quad \frac{a+1}{4} < 0 \quad \& \quad 2 \ge 0$

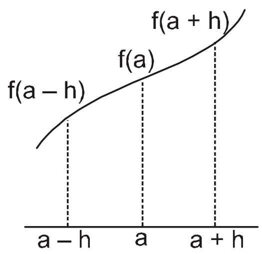
$\Rightarrow \quad a \in (-\infty, -5] \cup [3, \infty) \quad \& \quad a < -1 \quad \& \quad a \in R$

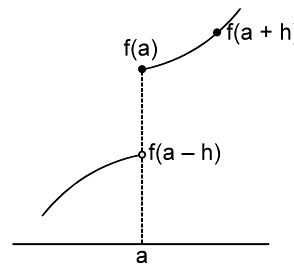
$\Rightarrow \quad a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (-\infty, 3]$

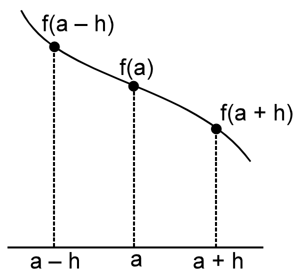
Monotonicity of function about a point :

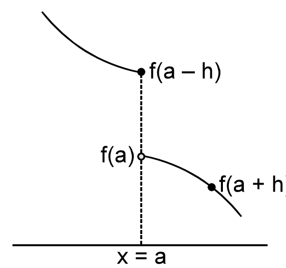
(i) A function f(x) is called as a strictly increasing function about a point (or at a point) $a \in D_f$ if it is strictly increasing in an open interval containing a (as shown in figure).



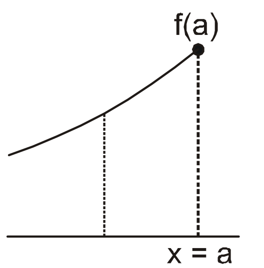


(ii) A function f(x) is called a strictly decreasing function about a point x = a, if it is strictly decreasing in an open interval containing a (as shown in figure).

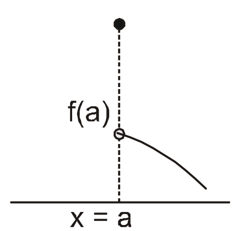




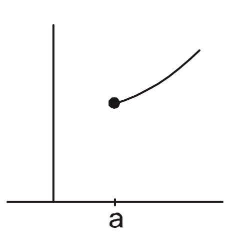
Note : If x = a is a boundary point then use the appropriate one sided inequality to test monotonicity of f(x).



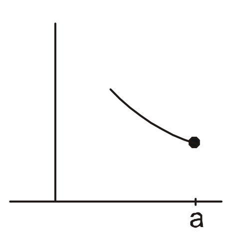
f is increasing at x = a



f is decreasing at x = a



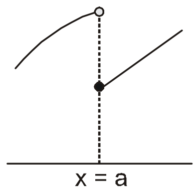
f is increasing at x = a



f is decreasing at x = a

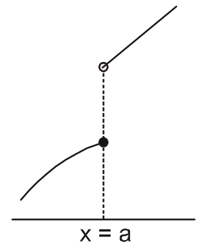
e.g. : Which of the following functions (as shown in figure) is increasing, decreasing or neither increasing nor decreasing at $x = a$.

(i)



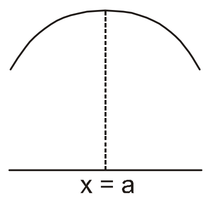
neither increasing nor decreasing at x = a

(ii)



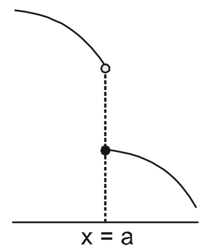
Increasing at x = a

(iii)



neither increasing nor decreasing at x = a

(iv)



decreasing at x = a

Test for increasing and decreasing functions about a point :

Let f(x) be differentiable.

(i) If $f'(a) > 0$ then f(x) is increasing at $x = a$.

(ii) If $f'(a) < 0$ then f(x) is decreasing at $x = a$.

(iii) If $f'(a) = 0$ then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a.

(a) If $f'(x)$ is positive on both the neighbourhoods, then f is increasing at $x = a$.

(b) If $f'(x)$ is negative on both the neighbourhoods, then f is decreasing at $x = a$.

(c) If $f'(x)$ have opposite signs on these neighbourhoods, then f is non-monotonic at $x = a$.

Ex.1 Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points $x = 0, 1, 2$.

Sol. $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

(i) $f'(0) = -3$ $\Rightarrow$ decreasing at $x = 0$

(ii) $f'(1) = 0$

also, $f'(x)$ is positive on left neighbourhood and $f'(x)$ is negative in right neighbourhood.

$\Rightarrow$ neither increasing nor decreasing at $x = 1$.

(iii) $f'(2) = 9$ $\Rightarrow$ increasing at $x = 2$

Note : Above method is applicable only for functions those are continuous at $x = a$.