CONTINUITY AND DIFFERENTIABILITY

SECOND ORDER DERIVATIVE

Second Order Derivatives: The concept of second order derivatives is not new to us. Simply put, it is the derivative of the first order derivative of the given function. However, it is important to understand its significance with respect to a function.

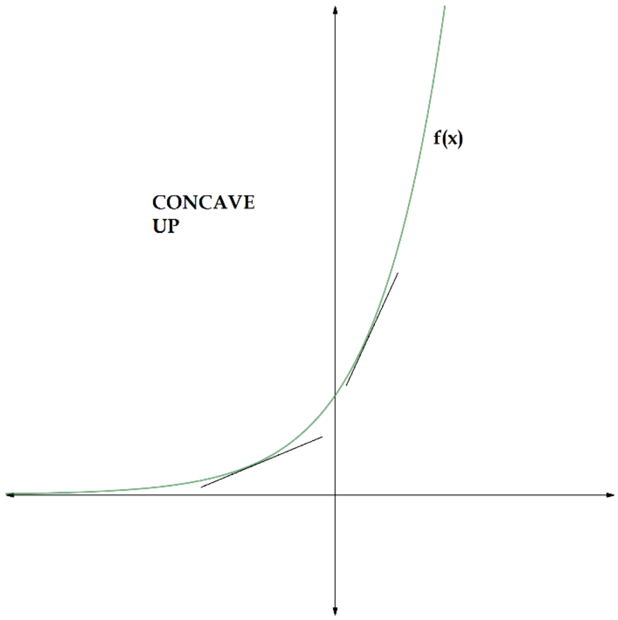
Similarly, as the First Order Derivative at a point gives us the slope of the tangent at that point or the instantaneous rate of change of the function at that point, the Second Order Derivatives are used to get an idea of the shape of the graph of a given function. They classify the behavior of a function in terms of concavity. Let us first introduce you to this new concept!

Concavity of a Function

Let f(x) be a differentiable function in a suitable interval. Then, the graph of f(x) can be categorized as –

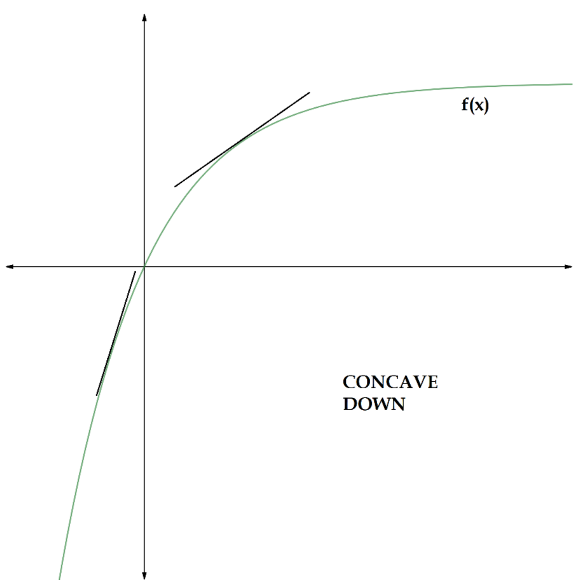
Concave Up

If the derivative $\left(\frac{d^2 f}{dx^2}\right)_{x=c} > 0$, the function is said to be Concave Up, or simply Concave, at the point (c, f(c)). In such a case, the points on the graph of the function in the neighborhood of c lie above the straight line which is tangent at the point (c, f(c)). The following figure should give you a better idea of what we mean here –



Concave Down

If the derivative $\left(\frac{d^2 f}{dx^2}\right)_{x=c} < 0$, the function is said to be Concave Down, or simply Convex, at the point (c, f(c)). In such a case, the points on the graph of the function in the neighborhood of c lie below the straight line which is tangent at the point (c, f(c)). The corresponding figure is as follows –

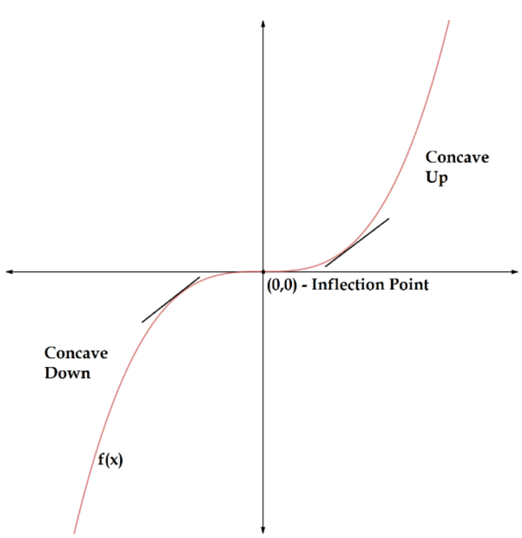


Important Remarks

In both of the figures above, you can ascertain that the function is increasing in the given interval. Since we know that if the derivative $\left(\frac{df}{dx}\right)_{x=c} > 0$, the function is increasing at the point (c, f(c)). Clearly, the first derivative of the function is greater than 0 for both functions represented above; and this information is insufficient to determine the shape of the function. We must know, how does the derivative of the function changes with changing x. If the derivative behaves as an increasing function i.e. $\frac{d}{dx}\left(\frac{df}{dx}\right) > 0$, the function is said to be Concave Up. And, if the derivative behaves as an decreasing function i.e. $\frac{d}{dx}\left(\frac{df}{dx}\right) < 0$, the function is said to be Concave Down. Therefore, the value of the second derivative is crucial in determining the shape of the graph of the function.

Points of Inflection

If the derivative $\left(\frac{d^2 f}{dx^2}\right)_{x=c} = 0$, the point (c, f(c)) is said to be a point of inflection of the given function. At such a point, the Concavity of the function changes its direction i.e. If the function is Concave Up, it becomes Concave Down, and vice-versa.



Now go through the solved example to understand the aforementioned concepts better.

Note:

We had already encountered another very important application of the Second Order Derivatives when we used them in the determination of Maxims and Minimas of a function. The test proceeds as follows –

If $f'(x) = 0$ at a given point $(c, f(c))$, then the second-order derivative at that point is computed. If $f''(x) > 0$ at that point, then it is a Local Minimum, and if $f''(x) < 0$ at that point, then it is a Local Maximum. If $f''(x) = 0$, higher order derivatives, or other methods of determination are required.

Now that we know the uses of the second-order derivatives, it is important to know how to calculate them for certain cases. The calculation very straightforward for functions of the form f(x), but it could become tricky, especially if the functions are given in the parametric form. The upcoming section explains how to proceed in such cases.

Second Order Derivatives of a Function in Parametric Form

We know that the first derivative of a function y(t) with respect to x(t), in Parametric Form can be directly calculated as –

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

One may then expect that the second derivative can then be given as –

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$$

But, this is a Wrong Argument. The Correct Method to calculate the second derivative, is to proceed by noting that now you are going to be differentiating the function dy/dx, which is a function of the paramter t, with respect to x(t). Thus, the process is as follows –

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

Ex.1 Given a function $f(x) = x^3 - 3x^2 + x - 2$. Determine the regions in the domain of f(x) where it is Concave Up, and Concave Down.

Sol. Since the given function f(x) is a polynomial function, the domain of f(x) is the set of all Real Numbers. Let us begin by calculating the first derivative of f(x)

$$\frac{df}{dx} = \frac{d}{dx}\left(x^3 - 3x^2 + x - 2\right)$$

$$\frac{df}{dx} = 3x^2 - 6x + 1$$

To determine Concavity, we need the second derivative as well. It can be calculated as follows –

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$\frac{d^2 f}{dx^2} = 6x - 6$$

$$= 6(x-1)$$

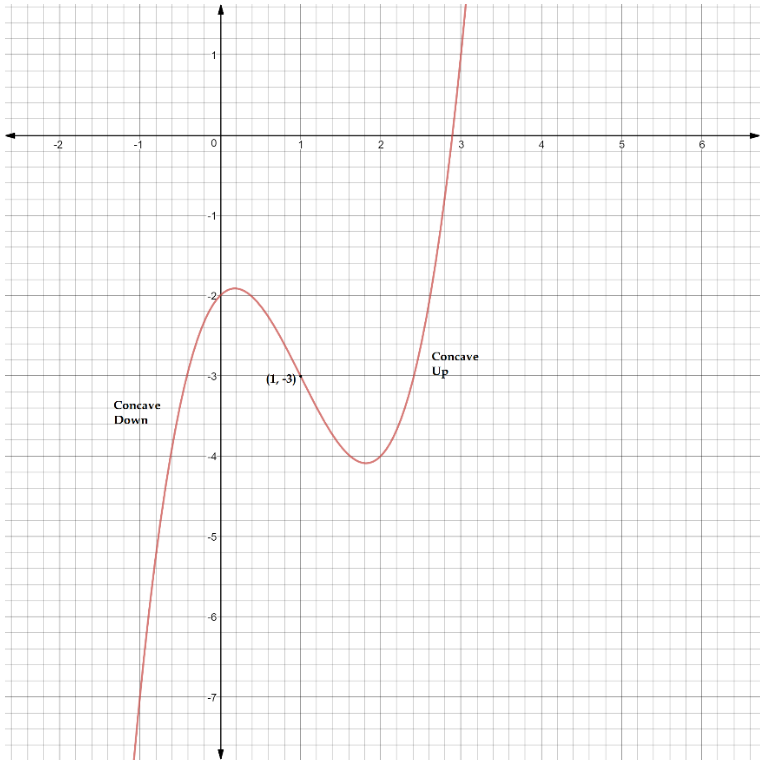
Three regions can be defined –

- x > 1
  For all x > 1, [f''(x) = 6(x – 1)] > 0.
  The function is Concave Up.

- x < 1
  For all x < 1, [f''(x) = 6(x – 1)] < 0.
  The function is Concave Down.

- x = 1
  [f''(x) = 6(x – 1)] = 0.
  Thus, (1, f(x = 1)) is a Point of Inflection of the given function.

Look at the graph of the function below and identify these regions yourselves.



Ex.2 Find the second derivative of y with respect to x, if $y = t^3$ and $x = \cos 2t$.

Sol. Using the formulae for parametric differentiation, The First Derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(t^3\right)}{\frac{d}{dt}(\cos 2t)}$$

$$= \frac{3t^2}{-2\sin 2t}$$

The Second Derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2}{-2\sin 2t}\right)}{\frac{d}{dt}(\cos 2t)}$$

Using the Quotient Rule to solve the derivative in the numerator –

$$\frac{d}{dt}\left(\frac{3t^2}{-2\sin 2t}\right) = \frac{\left[\frac{d}{dt}(-2\sin 2t)\right]\cdot 3t^2 - \left[\frac{d}{dt}\left(3t^2\right)\right]\cdot(-2\sin 2t)}{(-2\sin 2t)^2}$$

$$= \frac{(-4\cos 2t)\cdot\left(3t^2\right) - (6t)\cdot(-2\sin 2t)}{4\sin^2 2t}$$

$$= \frac{3t(\sin 2t - t\cos 2t)}{\sin^2 2t}$$

Combining this result with the derivative of the denominator above, we get the second derivative as –

$$\frac{d^2y}{dx^2} = \frac{\frac{3t(\sin 2t - tcos2t)}{\sin 2t}}{-2\sin 2t}$$

$$= \frac{3t(t\cos 2t - \sin 2t)}{2\sin^3 2t}$$

which is the required answer.

Similarly, the higher order derivatives for functions in parametric form can be computed by performing the process of parametric differentiation step by step.