CONTINUITY AND DIFFERENTIABILITY

MEAN VALUE THEOREM

The mean value theorem is a very important result in Real Analysis and is very useful for analyzing the behaviour of functions in higher mathematics. We'll just state the theorem directly first, before building it up logically as a general case of the Rolle's Theorem, and then understand its significance. So let's get to it!

The Mean Value Theorem

Let f(x) be a continuous function in a closed interval $x \in [a,b]$, and differentiable in the open interval $x \in (a,b)$ where $a<b$. Then the Mean Value Theorem says that –

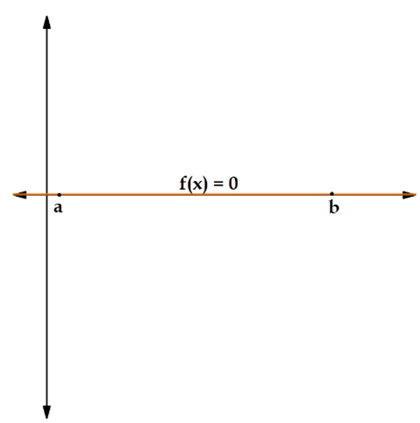
There is at least one point $x = c$ in the interval (a,b) such that $f'(c) = f(b)–f(a)/b–a$

The Logic Behind the Theorem

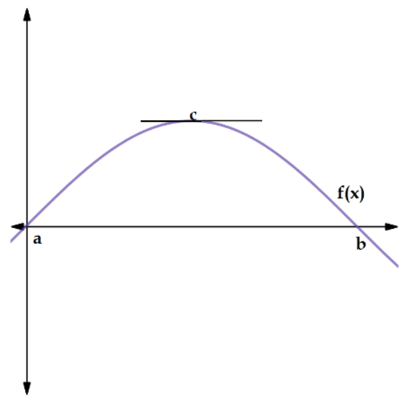
In two dimensional calculus, for a function f(x) which is continuous in a closed interval $x \in [a,b]$, and differentiable in the open interval $x \in (a,b)$ with $a<b$; if the condition $f(a) = f(b)$ is satisfied – Only a few specific types of behaviour of the function are allowed. They follow logically and are shown below –

For the sake of convenience, let $f(a) = f(b) = 0$ in our case.

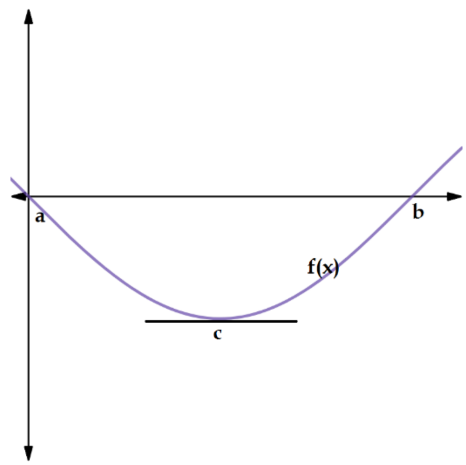
- $f(x) = 0 =$ constant.



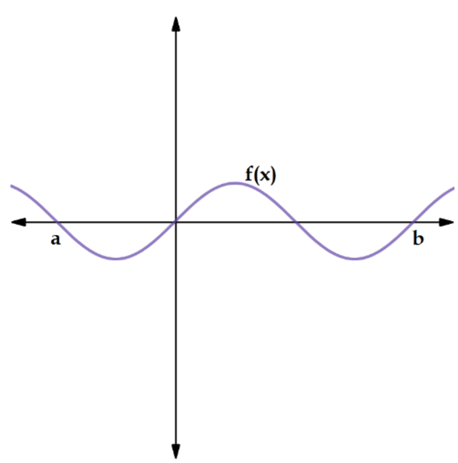
- f(x) first increases, then decreases.



- f(x) first decreases, then increases.



- f(x) increases/decreases multiple times.



In every case, you can notice at least one point on the function corresponding to an

x = c (marked on the graphs), where the function is stationary i.e. f'(c) = 0. If you put f(a) = f(b) in the statement of the Mean Value Theorem, you'll find that the theorem now states –

There is at least one point x = c in the interval (a,b) such that f'(c) = 0′.

This is the statement of the Rolle's Theorem. Thus, we see that the Mean Value Theorem is correct for the case of f(a) = f(b). Now let us prove the Mean Value Theorem for a given function.

Standard Proof of the Mean Value Theorem

Let us be given a function f(x) which is continuous in [a,b], and differentiable in (a,b). Let us also define a function g(x) which is the equation of the straight line passing through the points (a,f(a)) and (b,f(b)). From the two-point form of the equation for a straight line, we can derive g(x) as –

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

Using (a,f(a)) as (x_1,y_1) and (b,f(b)) as (x_2,y_2) –

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

Now let us define another function h(x) such that –

h(x) = f(x) – g(x)

$$= f(x) - \left[f(a) + \frac{f(b) - f(a)}{b-a} \cdot (x-a) \right]$$

Let us calculate the values of this function at the endpoints of the interval –

$$h(x=a) = f(a) - \left[f(a) + \frac{f(b) - f(a)}{b-a} \cdot (a-a) \right]$$

$$= 0$$

$$h(x=b) = f(b) - \left[f(a) + \frac{f(b) - f(a)}{b-a} \cdot (b-a) \right]$$

$$= f(b) - [f(a) + f(b) - f(a)]$$

$$= 0$$

Since, h(a) = h(b) = 0; the condition of applicability of the Rolle's Theorem is satisfied for the function h(x).

Thus, we are guaranteed the existence of a point x = c such that h'(c) = 0. Let us see what does this imply –

$$h'(c) = f'(c) - g'(c) = 0$$

$$f'(c) - \frac{d}{dx}\left(f(a) + \frac{f(b) - f(a)}{b-a} \cdot (x-a) \right)_{x=c} = 0$$

$$f'(c) - \frac{f(b) - f(a)}{b-a} = 0$$

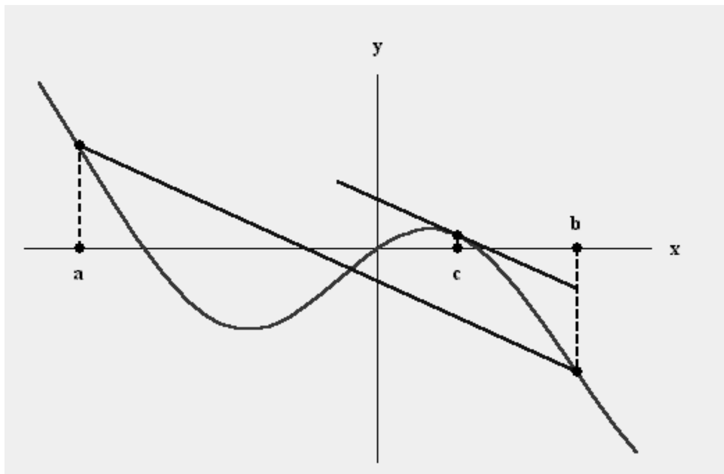
$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

where the final statement gives the result of the Mean Value Theorem.

The choice of a suitable function h(x) is arbitrary, and here we have chosen it for standardization purposes.

Interpretation of the Mean Value Theorem

The following interpretations can easily be understood by looking at the statement of the theorem:



- The theorem claims the existence of a point in the chosen interval, where the slope of the tangent is the same as the slope of the straight line joining the end-points of that interval.
- The theorem can be restated to claim the existence of a point in the chosen interval, where the instantaneous rate of change of the function is equal to its average rate of change over the entire interval.
- The theorem simply claims the existence of such a given point 'c' in the interval over which the function is continuous and differentiable. It doesn't provide any information as to the position of that point.

Let us now understand some important conclusions from the Mean Value Theorem before applying it to a problem for a better understanding.

Important Conclusions

⇒ **If f'(x) = 0 for all x in an interval (a,b), then f(x) is constant in the interval (a,b)**

Proof:

Given a function f(x) which is continuous in [a,b] and differentiable in (a,b). Initial assumption: f'(x) = 0. Now let us choose two points x_1 and x_2 in the interval (a,b) such that $x_1 < x_2$. Since the interval (x_1,x_2) is a subset of the interval (a,b), the function f(x) would be continuous and differentiable over this new interval as well.

Thus, the Mean Value Theorem is applicable here, and it claims the existence of a point c in (x_1,x_2) such that-

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

From the initial assumption, we have f'(x) = 0 for all x in (a,b). Thus, f'(c) = 0 necessarily. This implies-

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2) = f(x_1)$$

Since the choice of x_1 and x_2 is arbitrary, it can be concluded that $f(x_1) = f(x_2)$ for all x_1, x_2 in (a,b). This is only possible when f(x) = constant, in $x \in (a,b)$. Hence proved.

⇒ **If f'(x) = g'(x) for all x in an interval (a,b), then g(x) = f(x) + c.**

Proof:

Initial Assumption: f'(x) = g'(x). The result then follows from the previous one itself with just a slight manipulation. Define a new function –

$h(x) = f(x) – g(x)$

Since, f(x) and g(x) are continuous and differentiable in (a,b); we have –

$h'(x) = f'(x) – g'(x)$

But, $f'(x) = g'(x)$ from our initial assumption. Thus, from the previous result –

$h'(x) = 0$

$h(x) = c$ (some constant)

i.e. $g(x) – f(x) = c$

$g(x) = f(x) + c$

Hence Proved.

Ex.1 For a given function $f(x) = x^{4/3} – 3x^{1/3}$, find the value(s) of c in the interval (0,3), that satisfy the Mean Value Theorem.

Sol. To apply the condition of the Mean Value Theorem, we first need to find the derivative of the function in the given interval –

$$f(x) = x^{4/3} - 3x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - 3\cdot\frac{1}{3}x^{-2/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - x^{-2/3}$$

We also need the value of the function at the end-points –

$f(x = 0) = 0$

$$f(x=3) = 3^{4/3} - 3.3^{1/3}$$

$$= 3^{4/3} - 3^{4/3}$$

$$= 0$$

Note that since f(0) = f(3) here, the condition of Rolle's Theorem is satisfied.

Nevertheless, applying the Mean Value Theorem with 'c' as a point in the interval (0,3) –

$$f'(c) = \frac{f(3) - f(0)}{3 - 0}$$

$$f'(c) = 0$$

Using the expression for f'(x), we get –

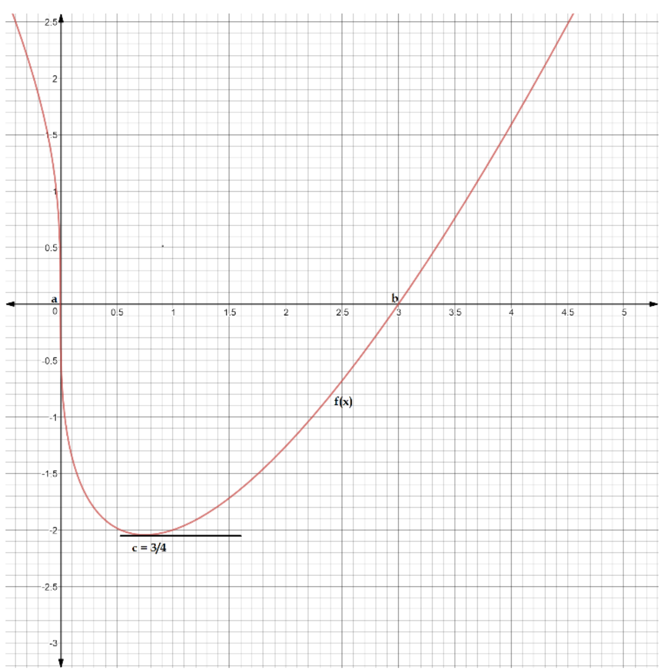
$$\frac{4}{3}c^{1/3} - c^{-2/3} = 0$$

$$\frac{4}{3}c^{1/3} = c^{-2/3}$$

$$c^{1/3+2/3} = \frac{3}{4}$$

c=34 ……. which is the required answer!

Look at the graph of the function below to better understand the behavior of the function in the given interval.



Ex.2 Who created the mean value theorem?

Sol. Augustin Louis Cauchy was the one to create the mean value theorem. Basically, Michel Rolle proved the restricted form of this theorem in 1691. It was proved only for polynomials without the techniques of calculus. Moreover, in 1823 Augustin stated and proved the mean value theorem in the modern form.

Ex.3 What does mean value theorem mean?

Sol. The Mean Value Theorem is one of the most essential theoretical tools in Calculus. It also says that if f(x) is definite and continuous on the interval [a,b] and differentiable on (a,b), in that case there is at least one number c in the interval (a,b) (that is $a < c < b$) such that.

Ex.4 Is the mean value theorem the same as the intermediate value theorem?

Sol. All the three are concerned with continuous functions on closed intervals. The Mean Value Theorem is all about differentiable functions and derivatives. On the other hand, the Intermediate Value theorem concerns continuous functions.

Ex.5 Who created Rolle's theorem?

Sol. Michel Rolle created Rolle's Theorem. He was a mathematician who created it in 1691. Further, this French mathematician was also the co-inventor in Europe of Gaussian elimination.