CONTINUITY AND DIFFERENTIABILITY

INTRODUCTION, CONTINUITY

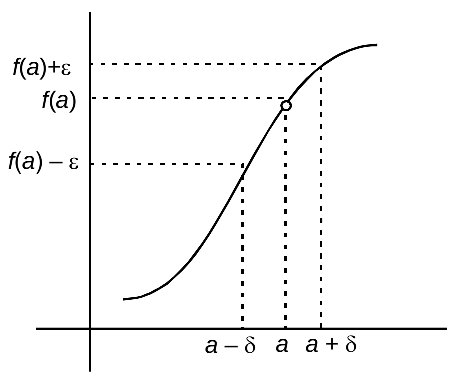
Continuity of a Function at a Point

A function f(x) is said to be continuous at a point x = a if and only if

$f(a^-) = f(a^+) = f(a)$ = finite number, i.e., $\lim_{x \to a} f(x)$ exists finitely, f(a) is a finite number and

$\lim_{x \to a} f(x) = f(a)$.

More precisely, for given e > 0 d > 0 such that 0 ≤ |x – a| < d ⇒ | f(x) – f(a) | < e



Ex.1 Consider the function

$$f(x) = \begin{cases} 5x - 4, & \text{if } 0 < x \le 1 \\ 4x^3 - 3x, & \text{if } 1 < x < 2 \end{cases}$$

Sol. At x = 1

$$\text{L.H.L.} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5x - 4)$$

Put x = 1 – h. As $x \to 1^-$, $h \to 0$

$$\therefore \quad \text{L.H.L.} = \lim_{h \to 0} [5(1-h) - 4] = 5(1-0) - 4 = 1$$

$$\text{R.H.L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(4x^3 - 3x\right)$$

Put $x = 1 + h$. As $x \to 1^+, h \to 0$

R.H.L. $= \lim_{h \to 0} \left[4(1+h)^3 - 3(1+h) \right]$

$= \left[4(1+0)^3 - 3(1+0) \right] = 1$

$f(1) = 5(1) - 4 = 1$

Since L.H.L. = R.H.L. = $f(1)$,

$\therefore f(x)$ is continuous at $x = 1$.

Continuity of a Function in Open Interval

A function is said to be continuous in an open interval (a, b), if it is continuous at each point of (a, b).

Continuity in Closed Interval

A function f(x) is said to be continuous on a closed interval [a, b] if

1. f(x) is continuous from right at x = a, i.e.,

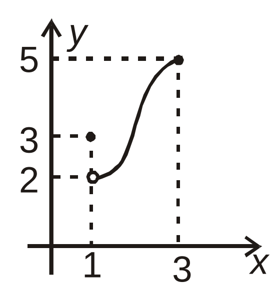
$\lim_{h \to 0} f(a+h) = f(a)$

2. f(x) is continuous from left at x = b, i.e.,

$\lim_{h \to 0} f(b-h) = f(b)$

3. f(x) is continuous at each point of the open interval (a, b)

Ex.2 The graph of a function f(x) is given.



Which of the following statements is not correct?

(a) f(x) is continuous on (1, 3)

(b) f(x) is continuous on (1, 3]

(c) f(x) is continuous on [1, 3]

(d) None of these

Sol. At x = 1

$f(1)=3, \lim_{x\to 1^+} f(x)=2$

f(x) is not continuous from right

At x = 3

$f(3)=5, \lim_{x\to 3^-} f(x)=5$

f(x) is continuous from left

At all the points is continuous because there is no jump in the graph.

Hence statement of option (3) is incorrect.

The nest illustration offers an example of piece - wise continuous function.

Ex.3 If $f(x)=\begin{cases} 1-2x & x<0 \\ 2 & x=0 \\ x^2+2 & x>0 \end{cases}$, then at x = 0

(1) f is continuous

(2) f is continuous from left

(3) f is continuous from right

(4) f has removable discontinuity

Sol. At x = 0

$LHL=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}(1-2(-h))=\lim_{h\to 0}(1+2h)=1$

$RHL=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}(0+h)^2+2=2$

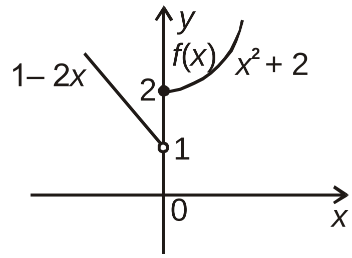
f(0) = 2

LHL $\neq$ RHL = f(0)

$\Rightarrow$ f is continuous from right but discontinuous from left.

only (3) is correct

Graphically:



Properties of Continuous Function

1. If f and g are continuous at $x = a$, then

(a) $f + g$ is continuous at $x = a$

(b) $f - g$ is continuous at $x = a$

(c) fg is continuous at $x = a$

(d) f / g is continuous at $x = a$, provided $g(a) \neq 0$

(e) kf is continuous at $x = a$, where k is any real constant

(f) $[f(x)]^{m/n}$ is continuous at $x = a$, provided $[f(x)]^{m/n}$ is defined on an interval containing a, and m, n are integers

2. If f is continuous at a and g is continuous at f(a) then gof is continuous at a

3. If f is continuous at $x = a$ and g is discontinuous at $x = a$, then $f + g$ and $f - g$ are discontinuous at $x = a$, whereas fg may be continuous at $x = a$.

4. If f is continuous at $x = a$ and $f(a) \neq 0$, then there exists an open interval $(a - d, a + d)$ such that has the same sign as f(a)

5. If f is a continuous function defined on [a, b] such that f(a). f(b) < 0, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

6. If f is a continuous function defined on [a, b] and k is any real number between f(a) and f(b), then there exists atleast one solution of the equation $f(x) = k$ in the open interval (a, b).

7. If a function f is continuous on a closed interval [a, b], then it is bounded on [a, b] i.e., there exists real number k and K such that $k \le f(x)$ for all $x \in [a, b]$
8. Every polynomial is continuous at every point of the real line.
9. Every rational function is continuous at every point where its denominator is different from zero.
10. Logarithmic functions, Exponential functions, Trigonometric functions, Inverse circular function and Absolute value functions are continuous in their domain of definition.

Ex.4 f + g may be a continuous function if

(1) f is continuous and g is discontinuous

(2) f is discontinuous and g is continuous

(3) f and g both are discontinuous

(4) None of these

Sol. Consider h(x) = f(x) + g(x)

If f(x) is continuous & g(x) is discontinuous, then let us assume that h(x) is continuous.

Now, $g(x) = \underset{\text{cont.fnn}}{h(x)} - \underset{\text{cont.fn}}{f(x)} \Rightarrow g(x)$ is a continuous function (by property 1(b) above)

Which is contradictory to the given fact that g(x) is a discontinuous function.

Hence our assumption that h(x) is continuous is wrong.

i.e., if f is continuous and g is discontinuous then f + g can't be a continuous function.

i.e., (1) is wrong.

Similarly (2) is wrong.

For (3), we take

$f(x) = x - [x]$ (Discontinuous function)

$g(x) = x + [x]$ (Discontinuous function)

$(f + g)(x) = (x - [x]) + (x + [x]) = 2x$ (Continuous function)

(3) is correct.

DISCONTINUITY OF FUNCTIONS

A function f(x), which is not continuous at a point x = a, is said to be discontinuous at that point.

The discontinuity may arise due to any of the following reasons

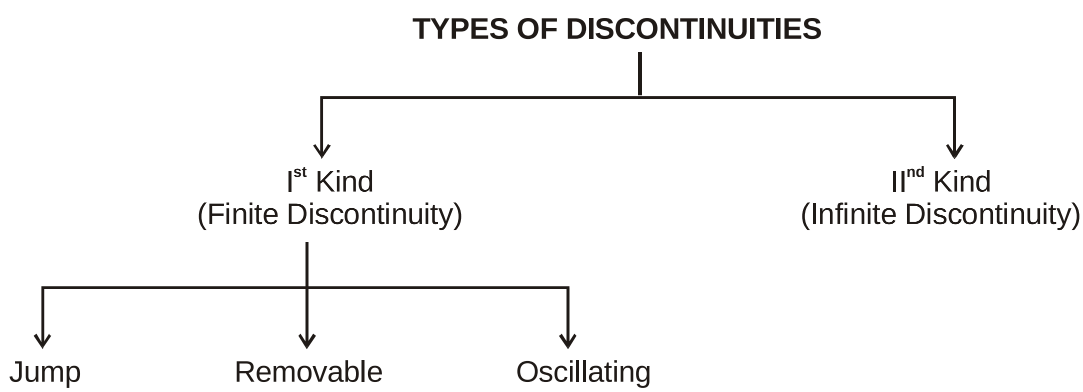
1. $\lim_{h\to0} f(a-h) \neq \lim_{h\to0} f(a+h)$, i.e., LHL and RHL exist, but are not equal

2 $\lim_{h\to0} f(a-h) = \lim_{h\to0} f(a+h) \neq f(a)$, LHL and RHL exist and are equal, but are different from $f(a)$.

3. $f(a)$ is not defined.

4. At least one of the limits $\lim_{h\to0} f(a-h)$ or $\lim_{h\to0} f(a+h)$ does not exist or at least one of these limits ∞ or $-\infty$.

Types of Discontinuities



(when a function keeps on oscillating as x $\to$ some number, f(x) doesn't come closer & closer to any number)

Removable Discontinuity

If $\lim_{h\to0} f(a-h)$ and $\lim_{h\to0} f(a+h)$ exist and are equal, but are not equal to f(a), then the Function f(x) is said to have a removable discontinuity at x = a. However, by suitably defining the function at x = a, f(x) can be made continuous at x = a.

Discontinuity of the First Kind

If $\lim_{h\to 0} f(a-h)$ and $\lim_{h\to 0} f(a+h)$ exist but are not equal, then the function f(x) is said to have a discontinuity of the first kind at x = a. Also called jump discontinuity.

If $\lim_{h\to 0} f(a-h)$ exists but not equal to f(a), then the function f(x) is said to have a discontinuity of the first kind from the left at x = a.

Similarly, if $\lim_{h\to 0} f(a+h)$ exists but not equal to f(a), then the function f(x) is said to have a discontinuity of the first kind from the right at x = a.

Discontinuity of the Second Kind

If at least one of the limits $\lim_{h\to 0} f(a-h)$ or $\lim_{h\to 0} f(a+h)$ does not exist or at least one of these limits is ∞ or ∞, then the function f(x) is said to have a discontinuity of the second kind at x = a.

If $\lim_{h\to 0} f(a-h)$ does not exist or is equal to ∞ or ∞, then the function f(x) is said to have a discontinuity of the second kind from the left at x = a. Discontinuity of the second kind from the right is similarly defined.

Ex.5 If $f(x)=\frac{1}{x^2-17x+66}$ then $f\left(\frac{2}{x-2}\right)$ is discontinuous at $x=$

(1) $2, \frac{7}{3}, \frac{25}{11}$

(2) $2, \frac{8}{3}, \frac{24}{11}$

(3) $2, \frac{7}{3}, \frac{24}{11}$

(4) None of these

Sol. Let $u=\frac{2}{x-2}$

$\because\ f(u)=\frac{1}{(u-6)(u-11)}$

$\therefore$ $f(u)$ is undefined when u is undefined, $u = 6, u = 11$ i.e., at $x = 2, \frac{2}{x-2} = 6, \frac{2}{x-2} = 11$

$x = 2, \quad x = \frac{7}{3}, \ x = \frac{24}{11}$

$\therefore$ (3) is correct

Ex.6 The function f(x) = |2 sgn 2x| + 2 has

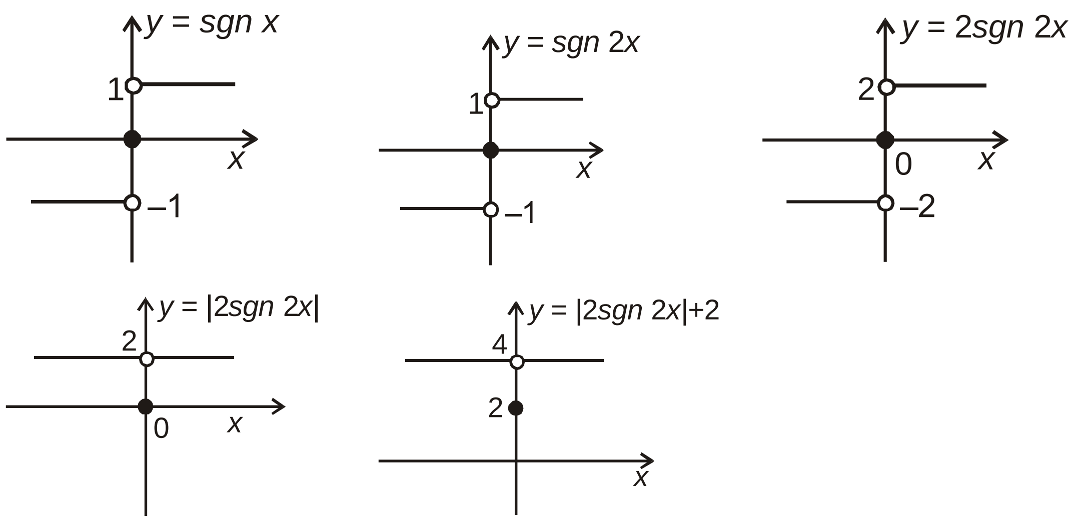
(1) Jump discontinuity

(2) Removable discontinuity

(3) Infinite discontinuity

(4) No discontinuity at x = 0

Sol. **Graphical Method**



At x = 0

LHL = RHL = 4 $\neq$ f(0) = 2

(2) is correct

Analytical Method :

To find RHL at x = 0, we put x = 0 + h (where h is small positive no.) and let h approach towards 0.

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \left| 2sgn \underbrace{2h}_{+ve} \right| + 2 = |2\times 1| + 2 = 4$$

Similarly, $LHL = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} |2\,\text{sgn}(2(-h))| + 2 = \lim\limits_{h \to 0} |2 \times (-1)| + 2 = 4$

$\text{f}(0) = |2 \text{ sgn } 2 \times 0| + 2 = |2 \times 0\,| + 2 = 2$

$\text{LHL} = \text{RHL} = 4\ {}^{1}\ \text{f}(0) = 2$

$\text{f}(\text{x})$ has removable discontinuity at $\text{x} = 0$