CONTINUITY AND DIFFERENTIABILITY

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The exponential and the logarithmic functions are perhaps the most important functions you'll encounter whenever dealing with a physical problem. They are the inverse of each other and can be used to represent a large range of numbers very conveniently.

They are continuous and differentiable over their entire domain, and the simplicity in notation, of their derivatives, would give you an idea about their huge significance in mathematics as well as other subjects. Let us now first understand these functions individually, before moving on to the connection between them.

Exponential Functions

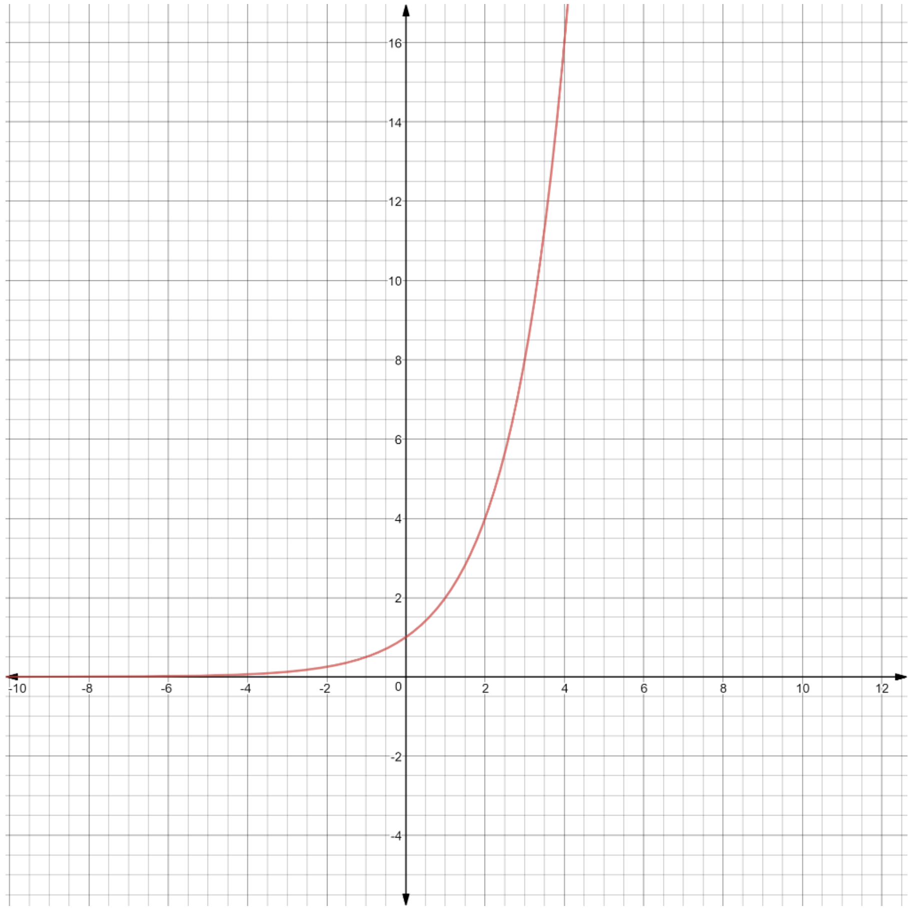
The term 'exponent' implies the 'power' of a number. For eg – the exponent of 2 in the number 2^3 is equal to 3. Clearly then, the exponential functions are those where the variable occurs as a power. An exponential function is defined as-

$$f(x)=a^x$$

where a is a positive real number, not equal to 1.
If a = 1, then $f(x) = 1^x$, which is equal to 1, ∀ x. Hence the graph of the function would just be a straight line of constant y (= 1). Depending on the value of 'a', we can have two possible cases:

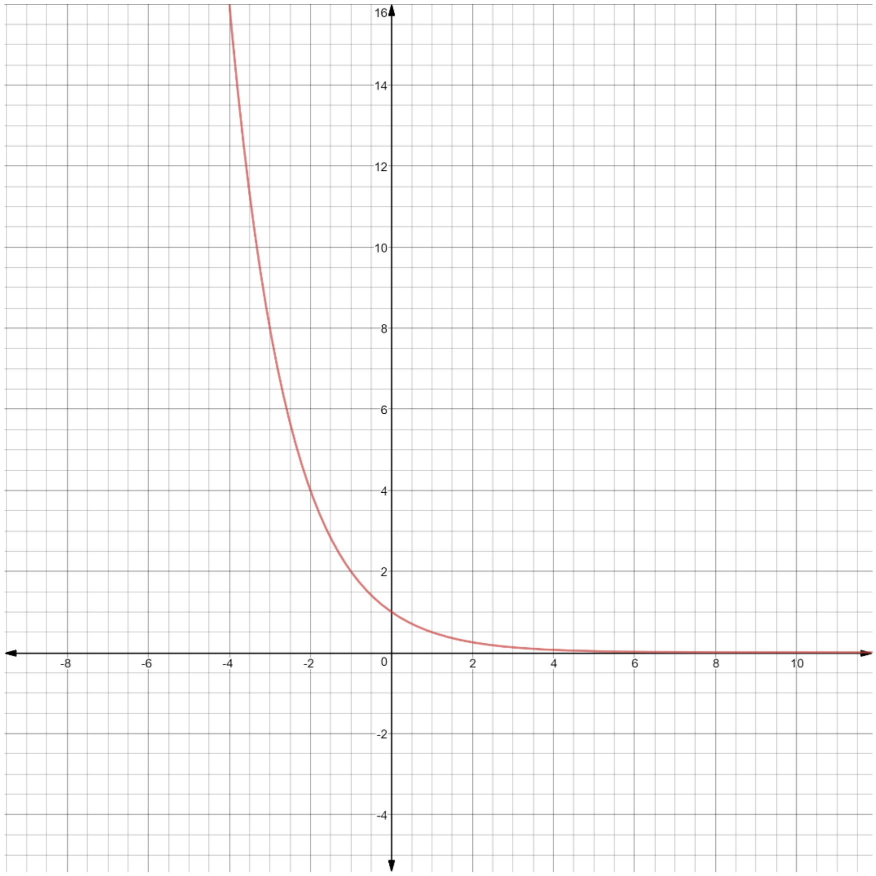
Case 1: $a > 1$

Here, the exponential function increases very rapidly with increasing x and tends to $+\infty$ as x tends to $+\infty$. When x = 0, $a^x = 1$; and when x tends to $-\infty$, the function tends to 0. The general graph of the function looks like this: (where a = 2)



Case 2: a < 1

The function decreases very rapidly with increasing x and tends to 0 as x tends to $+\infty$. When $x = 0$, $a^x = 1$ as usual; and when x tends to $-\infty$, the function tends to $+\infty$. The general graph of such a function looks like this – (where a = 2 again)



Properties of Exponential Functions

- The domain of the exponential function is (-∞,+∞) i.e. it is defined ∀ x.
- The range of the exponential function is (0,+∞). This property should be clear from the graph of the function a^x. Otherwise, also, it is logical that the power of any real number can't be a negative number. Only imaginary numbers can have such a behavior.
- The points (0,1) and (1, a) always lie on the graph of the function a^x.
- 'a' must necessarily be a positive number. If a is a negative number, then for any fractional values of x, we will get an imaginary number as a result which can't be plotted on the same graph. For eg- $(-2)^{1/2} = \sqrt{2}i$

- The Product Rule

$$a^x.a^y=a^{x+y}$$

- The Quotient Rule

$$\frac{a^x}{a^y}=a^{x-y}$$

- The exponential function is continuous and differentiable throughout its domain. The derivative is given as

$$\frac{d}{d(x)}(a^x)=a^x\ln(a)$$

where ln(a) or $\log_e(a)$is the natural logarithm of a. We'll define it formally in some time. The standard exponential function e^x is a unique function in mathematics with the property of being equal to its derivative. Thus, we have

$$\frac{d}{d(x)}(e^x)=e^x$$

In fact, the calculation behind these derivatives forms one of the methods of defining the number 'e' which is equal to 2.71828… That's all about exponential functions for now.

Logarithmic Functions

Since we had already disclosed that the logarithm function and the exponential function are inverses of each other, it should be obvious then that the logarithm function does the opposite of 'taking the power of a number'. Let's look at it mathematically –

General Notation

- Exponential Form –

$$b^y = x$$

- Logarithmic Form

$$y = \log_b^x$$

where ‘b’ is the base of the log.

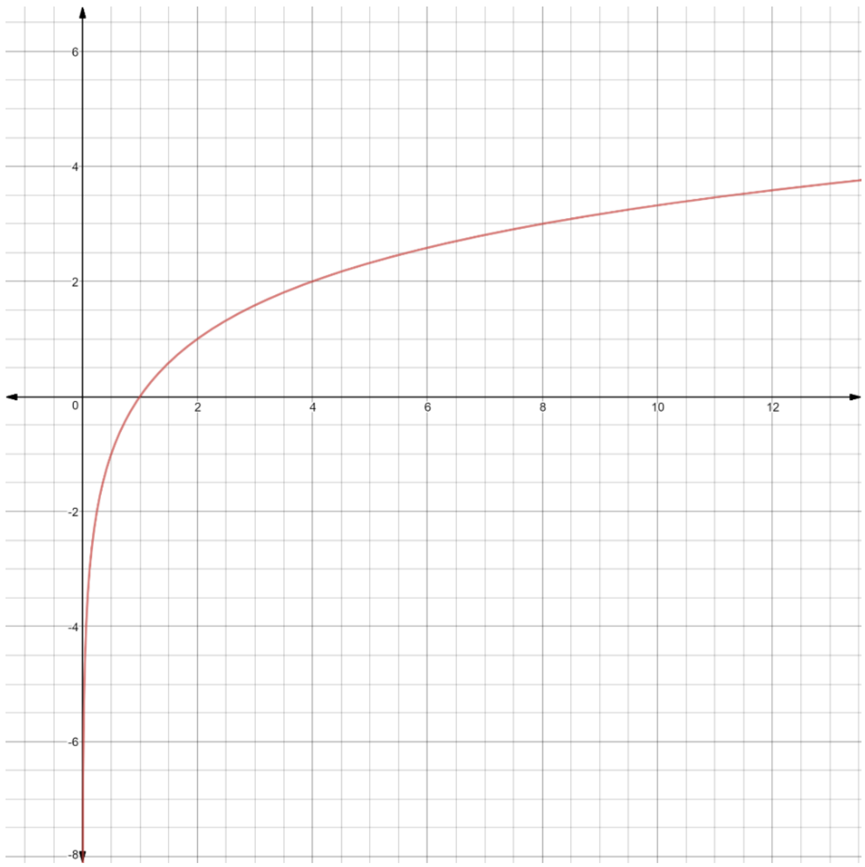
With these two forms, you can easily see that the value of the function $f(x) = \log_b x$ is the power to which ‘b’ must be raised to get ‘x’. ‘x’ therefore, can’t be negative since that would require ‘b’ to be imaginary, the conditions on the base ‘b’ –

- $b > 0$: It follows directly from the exponential representation of the logarithmic function.
- $b \neq 1$: Since 1 raised to any power would only give 1.

Depending on the value of ‘b’, we will have two possible cases –

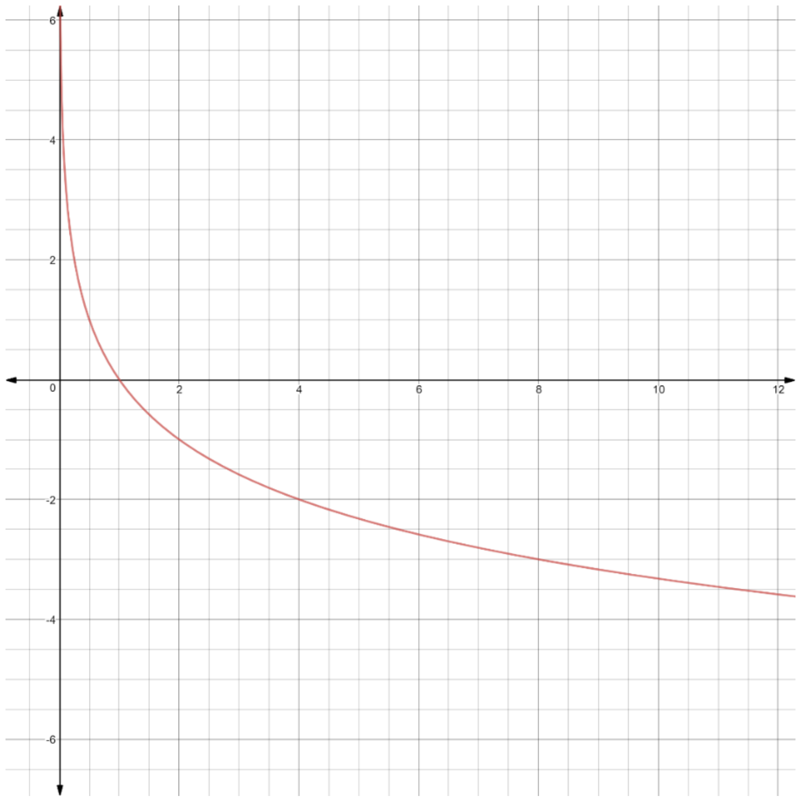
Case 1: $b > 1$

Here, the logarithmic function decreases very rapidly with decreasing x and tends to -∞ as x tends to 0. When x tends to +∞, the function also tends to +∞ with an ever-decreasing rate of increase. The general graph of the function looks like this – (where b = 2)



Case 2: 0 < b < 1

Here the function increases very rapidly to +∞ as x tends to 0, and falls at an ever decreasing rate to -∞ as x tends to +∞. The general graph is as shown – (where b = 0.5)



Properties of Logarithmic Functions

- The domain of the logarithmic functions is (0, +∞).
- The range of the logarithmic function is (-∞,+∞).
- The points (1,0) and (b,1) always lie on the graph of the function $\log_b x$.
- The Product Rule:

$$\log_b(xy) = \log_b x + \log_b y$$

- The Quotient Rule:

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

- The Power Rule:

$$\log_b a^x = x \log_b a$$

Generalization:

$$\log_b a^{f(x)} = f(x) \log_b a$$

- Change of Base Formula – To change the logarithm from a given base 'b' to base 'a'

$$\log_b x = \frac{\log_a x}{\log_a b}$$

- The logarithm function is continuous and differentiable throughout its domain. The derivative is given as

$$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x\ln(b)}$$

where ln(b) or $\log_e b$ is the natural logarithm of b. This is a standard logarithm function. It has the base = e = 2.71828. Its derivative –

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

since ln(e) = 1.

Relation between Exponential and Logarithmic Functions

We have already told you that the logarithmic and the exponential functions are inverses of each other. You can now verify this from the properties as well.

- The range and the domain of the two functions are exchanged.

- The points (0,1) and (1, a) always lie on the exponential function's graph while (1,0) and (b,1) always lie on the logarithmic function's graph.
- Product and Quotient Rules of the exponential and the logarithm functions follow from each other.

Let us now put our statement in a mathematical form for the standard functions –

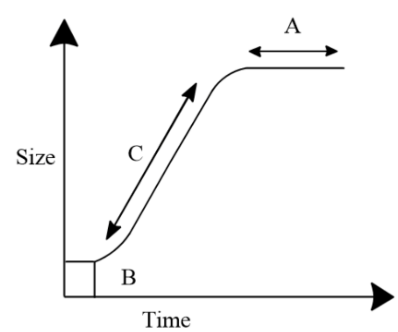
$$e^{\ln(x)} = \ln\left(e^{x}\right) = x$$

General formula –

$$b^{\log_b x} = \log_b b^{x} = x$$

Solved Examples for You

Ex.1 Given below is a graph drawn on the parameters of growth versus time.A,B,C respectively represents



- Exponential phase, log phase, and steady-state phase
- Steady-state phase, lag phase, and log phase
- Slow growing phase, lag phase, and steady-state phase
- Lag phase, steady-state phase, and logarithmic phase

- Log phase, lag phase, and steady-state phase

Sol. B. The first stage in the growth phase is a lag phase, where there is minimal growth. The next stage in the growth phase is the log phase, which is also known as the exponential phase where the growth is manifold. The final stage is a steady state where the growth is zero and thus known as the steady state.