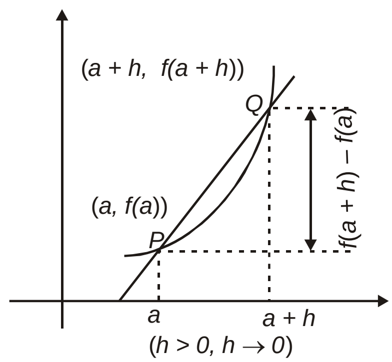
CONTINUITY AND DIFFERENTIABILITY

DIFFERENTIABILITY

DIFFERENTIABILITY OF FUNCTIONS

Differentiability at a Point



A function y = f(x) is said to be differentiable at a point a if at x = a left hand derivative f′ (a^-) and right hand derivative f¢ (a+) both exist finitely and are equal.
There common value is called derivative of f(x) at x = a.
Right hand derivative at x = a is defined as:

$$f'(a+) \equiv Lim_{h\to 0} \frac{f(a-h)-f(a)}{h} \,(h>0)$$

Left hand derivative at x = a :

$$f'(a-) \equiv Lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} \,(h>0)$$

Thus f(x) is differentiable at x = a if $f'(a^-) = f'(a^+)$ some fixed finite quantity.

Differentiability on an Interval

A function f(x) is said to be differentiable on an open interval (a, b) if f(x) is differentiable at every point of this interval (a, b).

It is differentiable on a closed interval [a, b] if it is differentiable on the open interval (a, b) and the limits

$$Rf'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a}$$

and $Lf'(b) = \lim_{h\to 0} \frac{f(a-h)-f(b)}{(b-h)-b}$ exist.

Properties of Differentiable Functions

1. Every polynomial function, exponential function and constant function is differentiable at each point of the real line.
2. Logarithmic functions, Trigonometric functions and Inverse Trigonometric functions are differentiable in their domain of definition.
3. The sum, difference, product and quotient of two differentiable functions is differentiable.
4. The composition of differentiable functions is a differentiable function.
5. If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.
6. If f(x) and g(x) both are not differentiable at a point, then the sum function f(x) + g(x) and the product function f(x). g(x) can still be differentiable at that point.

Ex.1 At the point x = 1, the function

$$f(x)=\begin{cases} x^3-1 & ; \quad 1<x<\infty \\ x-1 & ; \quad -\infty<x\le 1 \end{cases}$$

(1) Continuous and differentiable

(2) Continuous and non -differentiable

(3) Discontinuous and differentiable

(4) Discontinuous and non- differentiable

Sol. $RHL=\lim_{h\to 0} f(1+h)=\lim_{h\to 0}(1+h)^3-1$

$=\lim_{h\to 0}\left(1+3h+3h^2+h^3\right)-1=0$

$LHL=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}((1-h)-1)=0,\ f(1)=1-1=0$

$\because LHL=RHL=f(1) \quad \therefore f(x)$ is continuous at $x=1$.

Now, $f(x)$ may or may not be differentiable at $x=1$

$$RHD = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\left((1+h)^3-1\right)-(1-1)}{h}$$

$$= \lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h} = \lim_{h\to 0}\left(3+3h+h^2\right) = 3$$

$$LHD = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{((1-h)-1)-0}{-h}$$

$$= \lim_{h\to 0}\frac{(-h)}{(-h)} = 1$$

LHD ≠ RHD

f(x) is not differentiable at x = 1

(2) is correct.

Relation between Continuity and Differentiability

1. If a function f(x) is differentiable at a point x = a then it is continuous at x = a.
2. If f(x) is only continuous at a point x = a, there is no guarantee that f(x) is differentiable there.
3. If f(x) is not differentiable at x = a then it may or may not be continuous at x = a.
4. If f(x) is not continuous at x = a, then it is not differentiable at x = a.
5. If left hand derivative and right hand derivative of f(x) at x = a are finite (they may or may not be equal) then f(x) is continuous at x = a.

Differentiation of Basic Elementary Functions

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -cosec^2 x$

5. $\frac{d}{dx}(\sec x) = \sec x \tan x$

6. $\frac{d}{dx}(cosecx) = -cosecx \cot x$

7. $\frac{d}{dx}\left(e^x\right) = e^x$

8. $\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}(x > 0)$

9. $\frac{d}{dx}\left(a^x\right) = a^x(\ln a)(a > 0)$

10. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

11. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

12. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}, -\infty < x < \infty$

13. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}, -\infty < x < \infty$

14. $\frac{d}{dx}\left(cosec^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}, x \in R - [-1,1]$

15. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}, x \in R - [-1,1]$

Differentiation of Explicit Functions

1. Scalar product rule: $\frac{d}{dx}(cf(x)) = c'(x)$ (where c is a constant)

2. Sum and difference rule: $\frac{d}{dx}\{f(x) \pm g(x)\} = f'(x) \pm g'(x)$

3. Product rule: $\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$

4. Quotient rule: $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\}=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}\,(g(x)\neq 0)$

Ex.2 Find $\frac{dy}{dx}$ if

(a) $y=5x^{2/3}-3x^{5/2}+2x^{-3}$

(b) $y=x^3\tan^{-1}x$

(c) $y=\frac{\sin x+\cos x}{\sin x-\cos x}$

Sol. (a) $\frac{dy}{dx}=\frac{d}{dx}\left\{5x^{2/3}-3x^{5/2}+2x^{-3}\right\}$

$$=5\frac{d}{dx}\left(x^{2/3}\right)-3\frac{d}{dx}\left(x^{5/2}\right)+2\frac{d}{dx}\left(x^{-3}\right)$$

$$=5\cdot\frac{2}{3}x^{\frac{2}{3}-1}-3\cdot\frac{5}{2}x^{\frac{5}{2}-1}+2(-3)x^{-3-1}$$

$$=\frac{10}{3^3\sqrt{x}}-\frac{15}{2}x^{\frac{3}{2}}-\frac{6}{x^4}$$

(b) $\frac{dy}{dx}=\frac{d}{dx}\left(x^3\tan^{-1}x\right)=\left(\frac{d}{dx}x^3\right)\tan^{-1}x+x^3\frac{d}{dx}\left(\tan^{-1}x\right)$

$$=3x^2\tan^{-1}x+x^3\frac{1}{1+x^2}=3x^2\tan^{-1}x+\frac{x^3}{1+x^2}$$

(c) $\frac{dy}{dx}=\frac{\left(\sin x+\cos x\right)\left(\sin x-\cos x\right)-\left(\sin x+\cos x\right)\left(\sin x-\cos x\right)}{\left(\sin x-\cos x\right)^2}$

(‘’ denote differentiate w.r.t. x)

$$=\frac{(\cos x-\sin x)(\sin x-\cos x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}=\frac{-(1-2\sin x\cos x)-(1+2\sin x\cos x)}{(\sin x-\cos x)^2}=\frac{-2}{(\sin x-\cos x)^2}$$

Differentiation of composite functions: (Chain Rule)

If function h(x) is composition of two functions f(x) and g(x) as h(x) = f(g(x)). Now if h(x) = f(t) and t = g(x) then,

$$\frac{d}{dx}(h(x)) = \frac{dh(x)}{dt} \times \frac{dt}{dx} = \frac{df(t)}{dt} \times \frac{dg(x)}{dx} = f'(t)g'(x)$$

$$\Rightarrow \frac{d}{dx}(h(x)) = f'(g(x))g'(x)$$

This rule is called Chain Rule.

This rule can be extended to the compositions of any finite number of functions, for example, if y = f(g(h(x)))

Then, $\frac{dy}{dx} = f'(g(h(x)))g'(h(x))h'(x)$

Ex.3 Let $y = \log_e\left(\cos^3 x^4\right)$ find $\frac{dy}{dx}$.

Sol. Put $\cos^3 x^4 = t, \cos x^4 = z, x^4 = w$

then $\frac{dy}{dx} = \frac{d}{dt}\log t \times \frac{dz^3}{dz} \times \frac{d\cos w}{dw} \times \frac{dx^4}{dx}$

$= \frac{1}{t} \times 3z^2 \times (-\sin w) \times 4x^3$

$= \frac{12x^3 \cos^2\left(x^4\right)\sin\left(x^4\right)}{\cos^3\left(x^4\right)}$