DETERMINANTS

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(a) The value of a determinant remains unaltered, if the rows & columns are inter-changed,

e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

(b) If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only. e.g.

Let $\mathrm{D} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\mathrm{D}_1 = \begin{vmatrix} \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{a}_1 & \mathrm{b}_1 & c_1 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{vmatrix}$ Then $\mathrm{D}_1 = -\mathrm{D}$.

(c) If all the elements of a row (or column) are zero, then the value of the determinant is zero.

(d) If all the elements of any row (or column) are multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $\mathrm{D} = \begin{vmatrix} \mathrm{a}_1 & \mathrm{b}_1 & \mathrm{c}_1 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_3 \end{vmatrix}$ and $\mathrm{D}_1 = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ \mathrm{a}_2 & \mathrm{b}_2 & \mathrm{c}_2 \\ \mathrm{a}_3 & \mathrm{b}_3 & \mathrm{c}_3 \end{vmatrix}$ Then $\mathrm{D}_1 = KD$

(e) If all the elements of a row (or column) are proportional (or identical) to the element of any other row, then the determinant vanishes, i.e. its value is zero.

$$\text{e.g. If } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D = 0;$$

$$\text{If } D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow D_1 = 0$$

(f) If each element of any row (or column) is expressed as a sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

$$\text{e.g.} \begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note: that: If $D_r = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ a & b & c \end{vmatrix}$

where $r \in N$ and a,b,c, a_1, b_1,c_1 are constants, then

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(g) **Row - column operation :**

The value of a determinant remains unaltered under a column (C_i) operation of the form $C_i \to C_i + aC_j + bC_k$ (j, k $\neq$ i) or row (R_i) operation of the form $R_i \to R_i + aR_j + bR_k$ (j, k $\neq$ i). In other words, the value of a determinant is not altered by adding the elements of any row (or column) to the same multiples of the corresponding elements of any other row (or column) e.g. Let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D = \begin{vmatrix} a_1+\alpha a_2 & b_1+\alpha b_2 & c_1+\alpha c_2 \\ a_2 & b_2 & c_2 \\ a_3+\beta a_1 & b_3+\beta b_1 & c_3+\beta c_1 \end{vmatrix} \qquad (R_1 \to R_1+\alpha R_2; R_3 \to R_3+\beta R_2)$$

REMEMBER

(i) By using the operation $R_i \to xR_i + yR_j + zR_k$ ($j, k \neq i$), the value of the determinant becomes x times the original one.

(ii) While applying this property ATLEAST ONE ROW (OR COLUMN) must remain unchanged.

(h) Factor theorem :

If the elements of a determinant D are rational integral functions of x and two rows (or columns) become identical when $x = a$ then $(x - a)$ is a factor of D.

Note: that if r rows become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of D.

Ex.1 Simplify $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

Sol. Given detereminant is equal to

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} a+b & b+c & c^2 \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (a - b)(b - c)[ab^2 + abc + ac^2 + b^3 + b^2C + bc^2 - a^2b - a^2c - ab^2 - abc - b^3 - b^2c]$

$= (a - b)(b - c)[c(ab + bc + ca) - a(ab + bc + ca)]$

$= (a - b)(b - c)(c - a)(ab + bc + ca)$

Ex.2 Find the value of the determinant $\begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{vmatrix}$

Sol. $D = \begin{vmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & b & c^2 \end{vmatrix} = a\begin{vmatrix} a & b & c \\ ab & b^2 & bc \\ ac & b & c^2 \end{vmatrix} = abc\begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0$

Since all rows are same, hence value of the determinant is zero.

Ex.3 Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Sol. Let $D = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$ to D, we get

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1, we obtain

$$\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2\,b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$= 2\,c\,(a\,b + b^2 - bc) - 2\,b\,(b\,c - c^2 - ac)$

$= 2\,a\,b\,c + 2\,cb^2 - 2\,bc^2 - 2\,b^2c + 2\,bc^2 + 2\,abc$

$= 4\ abc$

Ex.4 Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Sol. $D = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$D = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \qquad (R_1 \to R_1 + R_2 + R_3)$

$D = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$D = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \qquad (C_3 \to C_3 - C_1; C_2 \to C_2 - C_1)$

$D = (a+b+c)^3$

Ex.5 Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Sol. We have

$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

(by property 5) $= 0 + 0 = 0$

(using property 3 and property 4)

Ex.6 If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is-

Sol. Applying $(C_3 \to C_3 - C_1)$

$$D = \begin{vmatrix} 3^2+k & 4^2 & 3 \\ 4^2+k & 5^2 & 4 \\ 5^2+k & 6^2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 9+k & 16 & 3 \\ 7 & 9 & 1 \\ 9 & 11 & 1 \end{vmatrix} = 0 \qquad (R_3 \to R_3 - R_2; R_2 \to R_2 - R_1)$$

$$\Rightarrow \quad k - 1 = 0$$

$$\Rightarrow \quad k = 1$$

Ex.7 Prove that $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = m(x-a)(x-b)$

Sol. using factor theorem put x = a

$$D = \begin{vmatrix} a & a & a \\ m & m & m \\ b & a & b \end{vmatrix} = 0$$

Since R_1 and R_2 are proportional which makes D = 0,

therefore (x – a) is a factor of D.

Similarly, by putting x = b, D becomes zero, therefore (x – b) is a factor of D.

$$D = \begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = \lambda(x-a)(x-b)$$

To get the value of λ, put $x = 0$ in equation (i)

$$\begin{vmatrix} a & a & 0 \\ m & m & m \\ b & 0 & b \end{vmatrix} = \lambda \cdot ab$$

$$amb = \lambda ab \quad \Rightarrow \quad \lambda = m$$

$$\therefore \quad D = m(x-a)(x-b)$$

MULTIPLICATION OF TWO DETERMINANTS

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1\ m_1 + b_1\ m_2 \\ a_2 l_1 + b_2 l_2 & a_2\ m_1 + b_2\ m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

(a) Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

(b) If D_1 is the determinant formed by replacing the elements of determinant D of order n by their corresponding cofactors then $D_1 = D^{n-1}$

Ex.8 Let a & b be the roots of equation $ax^2 + bx + c = 0$ and $S_n = a^n + b^n$ for n ³ 1.

Evaluate the value of the determinant . $\begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix}$

Sol. $$D = \begin{vmatrix} 3 & 1+S_1 & 1+S_2 \\ 1+S_1 & 1+S_2 & 1+S_3 \\ 1+S_2 & 1+S_3 & 1+S_4 \end{vmatrix} = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}^2$$

$$= [(1-\alpha)(1-\beta)(\alpha-\beta)]^2$$

$D = (a - b)^2 (a + b - ab - 1)^2$

Note: a & b are roots of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad \alpha + \beta = \frac{-b}{a} \& \alpha\beta = \frac{c}{a}$$

$$\Rightarrow \quad |\alpha - \beta| = \frac{\sqrt{b^2 - 4ac}}{|a|}$$

$$D = \frac{(b^2 - 4ac)}{a^2}\left(\frac{a+b+c}{a}\right)^2$$

$$=\frac{\left(b^2-4ac\right)(a+b+c)^2}{a^4}$$

Important Determinants

(i) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ \text{a} & \text{b} & \text{c} \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \text{a} & \text{b} & \text{c} \\ \text{a}^2 & \text{b}^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)\left(a^2+b^2+c^2-ab-bc-ca\right)$