DETERMINANTS

APPLICATION OF DETERMINANTS AND MATRICES

CRAMER'S RULE (SYSTEM OF LINEAR EQUATIONS)

- Simultaneous linear equations
  - Consistent (at least one solution)
    - Exactly one solution or Unique solution
      - Trivial solution
        - All variable zero is the only solution
      - Non trivial solution
        - At least one non zero variable satisfies the system
    - Infinite solutions
  - Inconsistent (no solution)

(a) Equations involving two variables

(i)	Consistent Equations	:	Definite & unique solution (Intersecting lines)
(ii)	Inconsistent Equations	:	No solution (Parallel lines)
(iii)	Dependent Equations	:	Infinite solutions (Identical lines)

Let, $a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$ then :

(1) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ $\Rightarrow$ Given equations are consistent with unique solution

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ $\Rightarrow$ Given equations are inconsistent

(3) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Given equations are consistent with infinite solutions

(b) Equations Involving Three variables

Let $a_1x + b_1y + c_1z = d_1$ ....(i)

$a_2x + b_2y + c_2z = d_2$ ....(ii)

$a_3x + b_3y + c_3z = d_3$ ....(iii)

Then, $x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

REMEMBER

(i) If $D \neq 0$ and at least one of $D_1, D_2, D_3 \neq 0$, then the given system of equations is consistent and has unique non trivial solution.

(ii) If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has trivial solution only.

(iii) If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent and has infinite solutions.

Note: that In case $\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_1x + b_1y + c_1z = d_2 \\ a_1x + b_1y + c_1z = d_3 \end{array}\right\}$ (Atleast two of d_1, d_2 & d_3 are not equal)

$D = D_1 = D_2 = D_3 = 0$. But these three equations represent three parallel planes. Hence the system is inconsistent.

(iv) If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution. .

(c) Homogeneous system of linear equations

Let $a_1x + b_1y + c_1z = 0$ ....(i)

$a_2x + b_2y + c_2z = 0$ ....(ii)

$a_3x + b_3y + c_3z = 0$ ....(iii)

$\Rightarrow$ $D_1 = D_2 = D_3 = 0$

The system always possesses at least one solution $x = 0, y = 0, z = 0$, which is called Trivial solution, i.e. this system is always consistent.

Check value of D

$D \neq 0$

Unique Trivial solution

$D = 0$

Trivial & Non-Trivial solutions

(infinite solutions)

Note: that if a given system of linear equations has Only Zero solutions for all its variables then the given equations are said to have TRIVIAL SOLUTION.

Also, note that if the system of equations $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$; $a_3x + b_3y + c_3 = 0$ is always consistent then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ but converse is not true.

Ex.1 Find the nature of solution for the given system of equations : $x + 2y + 3z = 1$; $2x + 3y + 4z = 3$; $3x + 4y + 5z = 0$

Sol. $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

Now $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 5$

$D = 0$ but $D_1 \neq 0$ Hence no solution.

Ex.2 Find the value of l, if the following equations are consistent :

x + y – 3 = 0; (1 + l)x + (2 + l)y – 8 = 0; x – (1 + l)y + (2 + l) = 0

Sol. The given equations in two unknowns are consistent, then D = 0

i.e. $\begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$

$\begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & 3\lambda-5 \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (5+\lambda) - (3\lambda-5)(-2-\lambda) = 0$

$\Rightarrow \quad \lambda \quad 3\lambda^2 + 2\lambda - 5 = 0$

$\therefore \quad \lambda = 1, -5/3$

Ex.3 If x, y, z are not all simultaneously equal to zero, satisfying the system of equations

sin(3q) x – y + z = 0; cos(2q)x + 4y + 3z = 0; 2x + 7y + 7z = 0, then find the values of q(0 ≤ q ≤ 2p).

Sol. Given system of equations is a system of homogeneous linear equations which posses non-zero solution set, therefore D = 0.

$\Rightarrow \quad D = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix}$

$$\Rightarrow \quad D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta & 4 & 7 \\ 2 & 7 & 14 \end{vmatrix} \qquad (C_3 \to C_3 + C_2)$$

$$D = \begin{vmatrix} \sin 3\theta & -1 & 0 \\ \cos 2\theta - 1 & 0.5 & 0 \\ 2 & 7 & 14 \end{vmatrix} \qquad \left(R_2 \to R_2 - \frac{R_3}{2}\right)$$

$$D = 14\left(\frac{\sin 3\theta}{2} + \cos 2\theta - 1\right)$$

D = 0

sin3q + 2cos2q – 2 = 0

⇒ 3sinq – 4sin^{3}q = 4sin^{2}q

⇒ (sinq)(4sin^{2}q + 4sinq – 3) = 0

⇒ (sinq)(2sinq – 1)(2sinq + 3) = 0

$$\Rightarrow \quad \sin\theta = 0; \sin\theta = \frac{1}{2}; \sin\theta = -\frac{5}{2}$$

$$\sin\theta = 0 \qquad \Rightarrow \theta = 0, \pi, 2\pi;$$

$$\sin\theta = \frac{1}{2} \qquad \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\sin\theta = -\frac{3}{2} \Rightarrow \text{ no solution}$$

$$\Rightarrow \quad \theta = 0, \frac{\pi}{6}, \frac{5\pi}{4}, \pi, 2\pi$$