DETERMINANTS

ADJOINT AND INVERSE OF A MATRIX

Cofactor matrix & adjoint matrix

Let $A = [a_{ij}]_n$ be a square matrix. The matrix obtained by replacing each element of A by corresponding cofactor is called as cofactor matrix of A, denoted as cofactor A. The transpose of cofactor matrix of A is called as adjoint of A, denoted as adj A.

i.e. if $A = [a_{ij}]_n$

then cofactor $A = [c_{ij}]_n$ when c_{ij} is the cofactor of a_{ij} " i & j.

Adj $A = [d_{ij}]_n$ where $d_{ij} = c_{ji}$ " i & j.

Properties of cofactor A and adj A

(a) $A\,.\text{adj } A = |A|\, I_n = (\text{adj } A)\, A$ where $A = [a_{ij}]_n$.

(b) $|\text{adj } A| = |A|^{n-1}$, where n is order of A.

In particular, for 3×3 matrix, $|\text{adj } A| = |A|^2$

(c) If A is a symmetric matrix, then adj A are also symmetric matrices.

(d) If A is singular, then adj A is also singular.

Ex.1 If $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 2 \\ 0 & -3 & 1 \end{bmatrix}$ and $B = [-1\ \ 2\ \ -1]$ find $B \times (\text{adj } A)$

Sol. The cofactors are

$c_{11} = +(4 + 6) = 10$ $\quad c_{12} = -(-1 - 0) = 1$

$c_{13} = +(3 - 0) = 3$ $\quad c_{21} = -(-1 + 9) = -8$

$c_{22} = +(2 - 0) = 2$ $\quad c_{23} = -(-6 + 0) = 6$

$c_{31} = +(-2 - 12) = -14$ $\quad c_{32} = -(4 + 3) = -7$

$c_{33} = +(8 - 1) = 7$

$$adjA = \begin{bmatrix} 10 & 1 & 3 \\ -8 & 2 & 6 \\ -14 & -7 & 7 \end{bmatrix}^T = \begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 7 \end{bmatrix}$$

Then $B \cdot (adjA) = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 7 \end{bmatrix}$

$$= [-10 + 2 - 3 \quad 8 + 4 - 6 \quad 14 - 14 - 7]$$

$$= [-11 \quad 6 \quad -7]$$

INVERSE OF A MATRIX (RECIPROCAL MATRIX)

Let A be a non-singular matrix. Then the matrix $\frac{1}{|A|}$ adj A is the multiplicative inverse of A (we call it inverse of A) and is denoted by A^{-1}.

We have A (adj A) = |A| I_n = (adj A) A

$\Rightarrow \quad A\left(\frac{1}{|A|} adjA\right) = I_n = \left(\frac{1}{|A|} adjA\right) A$ for A is non-singular

$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \text{ adj } A$

REMEMBER

(i) The necessary and sufficient condition for existence of inverse of A is that A is non-singular.

(ii) A^{-1} is always non-singular.

(iii) If A = dia (a_{11}, a_{22}, ….., a_{nn}) where $a_{ii} \neq 0$ " i, then A^{-1} = diag (a_{11}^{-1}, a_{22}^{-1}, …., a_{nn}^{-1}).

(iv) $(A^{-1})' = (A')^{-1}$ for any non-singular matrix A. Also adj (A′) = (adj A)′.

(v) $(A^{-1})^{-1} = A$ if A is non-singular.

(vi) Let k be a non-zero scalar & A be a non-singular matrix.

Then $(kA)^{-1} = \frac{1}{K} A^{-1}$.

(vii) $\left|A^{-1}\right| = \frac{1}{|A|}$ for $|A| \neq 0$

(viii) Let A be a non-singular matrix. Then AB = AC ⇒ B = C & BA = CA ⇒ B= C.

(ix) A is non-singular and symmetric ⇒ A^{-1} is symmetric.

(x) $(AB)^{-1} = B^{-1} A^{-1}$ if A and B are non- singular.

(xi) In general AB = 0 does not imply A = 0 or B = 0. But if A is non-singular and AB = 0, then B = 0.

Similarly B is non-singular and AB = 0 ⇒ A = 0. Therefore, AB = 0 ⇒ either both are singular or one of them is 0.

Ex.1 If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then verify that A adj A = | A | I. Also find A^{-1}

Sol. We have | A | = 1 (16 – 9) – 3 (4 – 3) + 3 (3 – 4) = 1 ≠ 0

Now

$A_{11} = 7$,

$A_{12} = -1$,

$A_{13} = -1$,

$A_{21} = -3$,

$A_{22} = 1$

$A_{23} = 0$

$A_{31} = -3$,

$A_{32} = 0$,

$A_{33} = 1$

Therefore adj $A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now $A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|.I$$

Also $A^{-1} \frac{1}{|A|} \text{ adj } A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Ex.2 Show that the matrix A = satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using the equation, find A^{-1} .

Sol. We have $A^2 = A \cdot A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore A A – 4A = – I

or $AA(A^{-1}) - 4\,A\,A^{-1} = -\,I\,A^{-1}$

(Post multiplying by A^{-1} because |A| ¹ 0)

or $A\,(A\,A^{-1}) - 4I = -\,A^{-1}$

or $AI - 4I = -\,A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Ex.3 Find the inverse of the matrix, $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

Sol. We have, $|A| = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = (14 - 12) = 12^{1} 0.$

So, A^{-1} exists.

The cofactors of the elements of |A| are given by

$A_{11} = 7,$ $A_{12} = (-4) = 4.$

$A_{21} = -(-3) = 3,$ $A_{22} = 2.$

$$(adjA) = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

Hence, $A^{-1} = \frac{1}{|A|} \cdot (\text{adj}A)$

$$= \frac{1}{2} \cdot \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{bmatrix}.$$