MATRICES

OPERATION ON MATRICES

ALGEBRA OF MATRICES

Addition : $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same order.

(A) Addition of matrices is commutative

i.e. $A + B = B + A$

(B) Matrix addition is associative

$(A + B) + C = A + (B + C)$

(C) Additive inverse

If $A + B = O = B + A$, then B is called additive inverse of A.

(D) Existence of additive identity

Let $A = [a_{ij}]$ be an $m \times n$ matrix and **O** be an $m \times n$ zero matrix, then $A + O = O + A = A$. In other words, **O** is the additive identity for matrix addition.

(E) Cancellation laws hold good in case of addition of matrices. If A,B,C are matrices of the same order, then $A + B = A + C \Rightarrow B = C$ (left cancellation law) and $B + A = C + A \Rightarrow B = C$ (right-cancellation law)

Note: The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Ex.1 If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 0 & 5 \\ 3 & 1 \end{bmatrix}$ and $A + B - D = O$ (zero matrix), then D matrix will be-

Sol. Let $D = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$

$$\therefore \quad A+B-D=\begin{bmatrix}1 & 3\\3 & 2\\2 & 5\end{bmatrix}+\begin{bmatrix}-1 & -2\\0 & 5\\3 & 1\end{bmatrix}-\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\Rightarrow\begin{bmatrix}1-1-a & 3-2-b\\3+0-c & 2+5-d\\2+3-e & 5+1-f\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix}$$

$\Rightarrow \quad -a = 0 \quad \Rightarrow \quad a = 0, \; 1 - b = 0 \;\Rightarrow\; b = 1,$

$3 - c = 0 \quad \Rightarrow \quad c = 3, \; 7 - d = 0 \;\Rightarrow\; d = 7,$

$5 - e = 0 \quad \Rightarrow \quad e = 5, \; 6 - f = 0 \;\Rightarrow\; f = 6$

$$D=\begin{bmatrix}0 & 1\\3 & 7\\5 & 6\end{bmatrix}.$$

MULTIPLICATION OF MATRIX BY SCALAR

Let l be a scalar (real or complex number) & $A = [a_{ij}]_{m \times n}$ be a matrix. Thus the product lA is
defined as $lA = [b_{ij}]_{m \times n}$ where $b_{ij} = la_{ij}$ " i & j.

$$\text{e.g. : } A=\begin{bmatrix}2 & -1 & 3 & 5\\0 & 2 & 1 & -3\\0 & 0 & -1 & -2\end{bmatrix} \& -3A \equiv (-3)A=\begin{bmatrix}-6 & 3 & -9 & -15\\0 & -6 & -3 & 9\\0 & 0 & 3 & 6\end{bmatrix}$$

Note: If A is a scalar matrix, then A = lI, where l is a diagonal entry of A

MULTIPLICATION OF MATRICES

Let A and B be two matrices such that the number of columns of A is same as number of rows of B.
i.e., $A = [a_{ij}]_{m \times p}$ & $B = [b_{ij}]_{p \times n}$.

Then $AB = [c_{ij}]_{m \times n}$ where $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$, which is the dot product of i^{th} row vector of A and j^{th} column vector of B.

e.g. : $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{bmatrix}, B=\begin{bmatrix}0 & 1 & 1 & 1\\ 0 & 0 & 1 & 0\\ 1 & 1 & 2 & 0\end{bmatrix}, AB=\begin{bmatrix}3 & 4 & 9 & 1\\ 1 & 3 & 7 & 2\end{bmatrix}$

(i) The product AB is defined if the number of columns of A is equal to the number of rows of B. A is called as premultiplier & B is called as post multiplier. AB is defined BA is defined.

(ii) In general $AB \neq BA$, even when both the products are defined.

(iii) $A(BC) = (AB)C$, whenever it is defined.

PROPERTIES OF MATRIX MULTIPLICATION

Consider all square matrices of order 'n'. Let M_n (F) denote the set of all square matrices of order n. (where F is Q, R or C). Then

(A) $A, B \in M_n(F) \Rightarrow AB \in M_n(F)$

(B) In general $AB \neq BA$

(C) $(AB)C = A(BC)$

(D) I_n, the identity matrix of order n, is the multiplicative identity.

$AI_n = A = I_n A$ " $A \in M_n(F)$

(E) For every non singular matrix A (i.e., $|A| \neq 0$) of M_n (F) there exist a unique (particular) matrix B Î M_n (F) so that $AB = I_n = BA$. In this case we say that A & B are multiplicative inverse of one another. In notations, we write $B = A^{-1}$ or $A = B^{-1}$.

(F) If l is a scalar (lA) B = l(AB) = A(lB).

(G) $A(B + C) = AB + AC$ " $A, B, C \in M_n(F)$

(H) $(A + B)C = AC + BC$ " $A, B, C \in M_n(F)$.

(i) Let $A = [a_{ij}]_{m \times n}$. Then $AI_n = A$ & $I_m A = A$, where I_n & I_m are identity matrices of order n & m respectively.

(ii) For a square matrix A, A^2 denotes AA, A^3 denotes AAA etc.

Ex.2 Find the value of x, if $\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = O$

Sol. We have $[1\ x\ 1]_{1\times 3}\begin{bmatrix}1 & 3 & 2\\ 2 & 5 & 1\\ 15 & 3 & 2\end{bmatrix}_{3\times 3}\begin{bmatrix}1\\ 2\\ x\end{bmatrix}_{3\times 1} = O = 0$

$\Rightarrow \quad [1 + 2x + 15 \quad 3 + 5x + 3 \quad 2 + x + 2]\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0$

$\Rightarrow \quad [16 + 2x \quad 6 + 5x \quad 4 + x]\begin{bmatrix}1\\ 2\\ x\end{bmatrix} = 0$

$\Rightarrow \quad [(16 + 2x) . 1 + (6 + 5x) . 2 + (4 + x) . x] = 0$

$\Rightarrow \quad (16 + 2x) + (12 + 10x) + (4x + x^2) = 0$

$\Rightarrow \quad x^2 + 16x + 28 = 0$

$\Rightarrow \quad (x + 14)(x + 2) = 0$

$\Rightarrow \quad x + 14 = 0 \quad$ or $\quad x + 2 = 0$

$\Rightarrow \quad x = -14 \quad$ or $\quad x = -2$

Hence, $x = -14$ or $x = -2$

POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX

For a square matrix A, $\mathbf{A^n} = \underbrace{A.A.A..........A}_{\text{upto n times}}$ where $n \in N$

REMEMBER

(i) $A^m . A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn}$, where $m,n \in N$

(ii) If A and B are square matrices of same order and AB = BA then

$(A + B)^n = {}^nC_0A^n + {}^nC_1A^{n-1}B + {}^nC_2A^{n-2}B^2 + ... + {}^nC_nB^n$

Note: that for a unit matrix I of any order , $I^m = I$ for all $m \in N$.

SPECIAL SQUARE MATRICES

(A) Idempotent Matrix:

A square matrix is idempotent provided $A^2 = A$.

For idempotent matrix note the following:

(i) $A^n = A \quad n \geq 2, n$ N.

(ii) determinant value of idempotent matrix is either 0 or 1

(B) Periodic Matrix :

A square matrix which satisfies the relation $A^{k+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.

Note: that period of an idempotent matrix is 1.

(C) Nilpotent Matrix :

A square matrix of the order 'n' is said to be nilpotent matrix of order m, $m \in N$,

if $A^m = O$ & $A^{m-1} \neq O$.

(D) Involutary Matrix :

If $A^2 = I$, the matrix is said to be an involutary matrix. i.e. square roots of identity matrix is involutary matrix.

Note: The determinant value of involutary matrix is 1 or –1.

Ex.3 Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

Sol. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^5 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-37 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$A^3 = O$ i.e., $A^k = O$

Here $k = 3$

Hence A is nilpotent of order 3.

Ex.4 Show that a square matrix A is involutory, if $(I - A)(I + A) = O$

Sol. Let A be involutory

Then $A^2 = I$

$(I - A)(I + A) = I^2 + IA - AI - A^2 = I + A - A - A^2 = I - A^2 = O$

Conversly, let $(I - A)(I + A) = O$

$\Rightarrow \quad I^2 + IA - AI - A^2 = O$

$\Rightarrow \quad I + A - A - A^2 = O$

$\Rightarrow \quad I - A^2 = O$

$\Rightarrow$ A is involutory