MATRICES

INVERTIBLE MATRICES

ORTHOGONAL MATRIX

A square matrix is said to be orthogonal matrix if A $A^{T} = I$

Note

(i) The determinant value of orthogonal matrix is either 1 or –1.

(ii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$\Rightarrow \quad A^{T} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$AA^{T} = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 & a_1c_1 + a_2c_2 + a_3c_3 \\ b_1a_1 + b_2a_2 + b_3a_3 & b_1^2 + b_2^2 + b_3^2 & b_1c_1 + b_2c_2 + b_3c_3 \\ c_1a_1 + c_2a_2 + c_3a_3 & c_1b_1 + c_2b_2 + c_3b_3 & c_1^2 + c_2^2 + c_3^2 \end{bmatrix}$$

If $AA^{T} = I$, then

$$\sum_{i=1}^{3} a_i^2 = \sum_{i=1}^{3} b_i^2 = \sum_{i=1}^{3} c_i^2 = 1 \text{ and } \sum_{i=1}^{3} a_i b_i = \sum_{i=1}^{3} b_i c_i = \sum_{i=1}^{3} c_i a_i = 0$$

Ex.1 Determine the values of a, b, g, when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Sol. $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

But given A is orthogonal.

$$AA^{\mathrm{T}} = \mathrm{I}$$

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$4b^2 + g^2 = 1$ ....**(i)**

$2b^2 - g^2 = 0$ ....**(ii)**

$a^2 + b^2 + g^2 = 1$ ....**(iii)**

From (i) and (ii), $6\beta^2 = 1 \quad \therefore \quad \beta^2 = \frac{1}{6}$ and $\gamma^2 = \frac{1}{3}$

From (iii) $\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$

Hence, $\alpha = \pm\frac{1}{\sqrt{2}}, \beta = \pm\frac{1}{\sqrt{6}}$ and $\gamma = \pm\frac{1}{\sqrt{3}}$

ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A, denoted by adj A, is defined as the transpose of the cofactor matrix.

Then, adjA = $[C_{ij}]^T$ $\Rightarrow$ $adjA = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

Theorem A (adj. A) = (adj. A).A = |A| I_n.

Proof: $A(adjA) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

$$\begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix} = |A| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ A.(Adj. A) $= |A| I$

(whatever may be the value only |A| will come out as a common element)

If $|A| \neq 0$ then $\dfrac{A(adj.A)}{|A|} = I =$ unit matrix of the same order as that of A

Properties of adjoint matrix

If A be a square matrix of order n, then

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) adj(adj A) = $|A|^{n-2}$ A, where $|A|^1$ 0

(iii) $adj(adjA)| = A|^{(i-1)^2}$, where $|A| \neq 0$

(iv) adj(AB) = (adj B) (adj A)

(v) adj(KA) = K^{n-1} (adj A) , K is a scalar

(vi) adj A^T = $(\text{adj } A)^T$

Method to find Adjoint of a 2 ×2 Square Matrix, Directly

Let A be a 2 × 2 square matrix. In order to find the adjoint simply interchange the diagonal elements and reverse the sign of off diagonal elements (rest of the elements).

e.g. If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow adjA = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

Ex.2 If $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$, then $adj(adj\text{A})$ is equal to -

Sol. $|\text{A}| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 8$

Now $adj(adj\text{A}) = |\text{A}|^{3-2}\ \text{A}$

$$= 8\begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{vmatrix} = 16\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Inverse of a Matrix (Reciprocal Matrix)

A square matrix A said to be invertible (non singular) if there exists a matrix B such that,

A B = I = B A

B is called the inverse (reciprocal) of A and is denoted by A^{-1}.

Thus $A^{-1} = B \Leftrightarrow A\,B = I = B\,A$.

We have, A . (adj A) $= \frac{1}{2} A \frac{1}{2} I_n$

A^{-1} A (adj A) = $A^{-1} I_n$ |A|

I_n (adj A) = $A^{-1}\ \frac{1}{2} A \frac{1}{2} I_n$

$$A^{-1} = \frac{(adj\text{A})}{|\text{A}|}$$

Note: The necessary and sufficient condition for a square matrix A to be invertible is that $\frac{1}{2} A \frac{1}{2} \neq 0$.

Imp. Theorem :

If A & B are invertible matrices of the same order , then $(AB)^{-1} = B^{-1}\ A^{-1}$. This is reversal law for inverse.

REMEMBER

(i) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(ii) If A is invertible,

(A) $(A^{-1})^{-1} = A$;

(B) $(A^k)^{-1} = (A^{-1})^k = A^{-k}, k \in N$

(iii) If A is an Orthogonal Matrix. $AA^T = I = A^TA$

(iv) A square matrix is said to be orthogonal if, $A^{-1} = A^T$.

(v) $|A^{-1}| = \frac{1}{|A|}$

Ex.3 Prove that if A is non-singular matrix such that A is symmetric then A^{-1} is also symmetric.

Sol. $A^T = A$ [$\because$ A is a symmetric matrix]

$(A^T)^{-1} = A^{-1}$ [since A is non-singular matrix]

$(A^{-1})^T = A^{-1}$ Hence proved

Ex.4 If $A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $M = AB$, then M^{-1} is equal to-

Sol. $M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}$

$|M| = 6, adj M = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

$\therefore \quad M^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/6 \end{bmatrix}$

Ex.5 Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using the equation, find A^{-1}. .

Sol. We have $A^2 = A \cdot A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Now $A^2 - 4A + I = 0$

Therefore A A – 4A = – I

Or $AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$

(Post multiplying by A^{-1} because $|A| \neq 0$)

or $A(AA^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Matrix Polynomial

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_nx^0$, then we define a matrix polynomial

$f(a) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots\ldots. + a_nI^n$.

where A is the given square matrix. If $f(a)$ is the null matrix, then A is called the zero or root of the polynomial $f(x)$.

System of Equation & Criterian For Consistency Gauss - Jordan Method

$x + y + z = 6 \qquad x - y + z = 2 \qquad 2x + y - z = 1$

$$\begin{pmatrix} x+y+z \\ x-y+z \\ 2x+y-z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$A X = B$

$\Rightarrow \quad A^{-1} A X = A^{-1} B$

$$X = A^{-1}B = \frac{(adj \cdot A) \cdot B}{|A|}.$$