MATRICES

ELEMENTARY OPERATION OF A MATRIX

Elementary Row Operations

There are many applications of elementary row operations. They can be used to solve a system of equations in an easy way, they can be used to find the rank of a matrix etc. The elementary row transformations are also used to find the inverse of a matrix A without using any formula like A^{-1} = (adj A) / (det A).

Let us see how to apply inverse row operations for performing multiple things in an easy way.

What are Elementary Row Operations?

While applying the elementary row operations, we usually represent the first row by R_1, the second row by R_2, and so on. There are primarily three types of elementary row operations:

- Interchanging two rows.
  For example, interchanging the first and second rows is shown by $R_1 \leftrightarrow R_2$.
- Multiplying/dividing a row by a scalar.
  For example, if the first row (all elements of the first row) is multiplied by some scalar, say 3, it is shown as $R_1 \to 3R_1$.
- Multiplying/dividing a row by some scalar and adding/subtracting to the corresponding elements of another row.
  For example, if the first row is multiplied by 3, and added to the second row, we can write it either as $R_1 \to 3R_1 + R_2$ (or) $R_2 \to R_2 + 3R_1$.

It is a common practice to write the same row on the left side of the arrow and on the very first occurrence of the right side of the arrow.

Elementary Row Operations to Solve a System of Equations

We can solve a system of equations written in matrix form AX = B, by writing the augmented matrix [A B] and applying the elementary row operations on it to convert it into the echelon form (preferably the upper triangular form). Applying all the above three row operations do not alter the augmented matrix as:

- Interchanging two rows is nothing but swapping two equations of the system and this doesn't affect the solution.
- Multiplying a row by some scalar does not alter the augmented matrix, as we always can multiply both sides of an equation by a scalar without affecting the equation.
- Multiplying one row by scalar and adding it to the other row is nothing but multiplying an equation by a scalar and adding it to another equation and we usually do this to solve a system of equations.

This process of applying row operations to solve a system is known as Gauss elimination. We can see an example of applying row transformations to solve a system of equations in the **"Elementary Row Operations Examples**" section below.

Elementary Row Operations to Find Inverse of a Matrix

To find the inverse of a square matrix A, we usually apply the formula, A^{-1} = (adj A) / (det A). But this process is lengthy as it involves many steps like calculating cofactor matrix, adjoint matrix, determinant, etc. To make this process easy, we can apply the elementary row operations. Here are the steps for doing the same.

- Consider the augmented matrix [A | I], where I is the identity matrix that is of the same order as A.
- Apply row transformations to convert the left side matrix A into I.
- Then the right side matrix (that replaces the original matrix I) is nothing but A^{-1}.

Elementary Row operations to find Determinant

Usually, we find the determinant of a matrix by finding the sum of the products of the elements of a row or a column and their corresponding cofactors. But this process is difficult if the terms of the matrix are expressions. But we can apply the elementary row operations to find the determinant easily. But some of the row operations affect the determinant in the following ways:

- Interchanging two rows of a determinant changes its sign.
- Multiplying a row by some scalar multiplies the determinant by the same scalar.
- Multiplying a row by some scalar and adding the result to another row doesn't alter the determinant.

Elementary Row Operations to Find Rank of a Matrix

The rank of a matrix is the number of linearly independent rows (or columns) in it. We can apply the elementary row operations on the matrix to find its rank in two ways:

- We can convert it into Echelon form and count the number of non-zero rows in it which gives its rank.
- We can convert it int the normal form $\begin{bmatrix} l_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r is the identity matrix of order r. Then the rank of the matrix = r.

Ex.1 Perform the following elementary row operations on the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \\ -4 & 0 & 2 \end{bmatrix}$

(a) $R_1 \leftrightarrow R_2$ (b) $R_2 \rightarrow R_2 - 5\, R_1$

Sol. (a) $R_1 \leftrightarrow R_2$ means swapping (or interchanging) the first two rows.

Then the result is $\begin{bmatrix} 3 & 2 & 0 \\ 1 & 2 & -1 \\ -4 & 0 & 2 \end{bmatrix}$

(b) We have R_1 (the first row) = [1 2 -1].

Then $-5R_1$ = [-5 -10 5].

$R_2 - 5R_1$ = [3 2 0] + [-5 -10 5] = [-2 -8 5]

$R_2 \to R_2 - 5R_1$ means replace R_2 by the row obained by doing $R_2 - 5R_1$. Then the resultant matrix is $\begin{bmatrix} 1 & 2 & -1 \\ -2 & -8 & 5 \\ -4 & 0 & 2 \end{bmatrix}$

Ex. 2 Solve the following system of equations using elementary row transformations:

$2x - y + 3z = 8$, $-x + 2y + z = 4$, and $3x + y - 4z = 0$.

Sol. The matrix equation of the given system is:

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

The augmented matrix is,

$$[AB] = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

Here we convert the last two elements of the first column (-1 and 3) to be zero. We use R_1 in this process.

Apply $R_2 \to 2R_2 + R_1$ and $R_3 \to 2R_3 - 3R_1$, we get:

$$= \begin{bmatrix} 2 & -1 & 3 & 8 \\ 0 & 3 & 5 & 16 \\ 0 & 5 & -17 & -24 \end{bmatrix}$$

We convert the last element of the second column (5) to be a zero. We use R_2 in this process.

Now, apply $R_3 \to 3R_3 - 5R_2$,

$$= \begin{bmatrix} 2 & -1 & 3 & 8 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -76 & -152 \end{bmatrix}$$

Now, we expand the above matrix as equations:

$2x - y + 3z = 8$ ... (1)

3y + 5z = 16 ... (2)

-76z = -152 ... (3)

From (3), z = (-152) / (-76) = 2.

From (2), 3y + 5(2) = 16 ⇒ 3y = 6 ⇒ y = 2.

From (1), 2x - 2 + 3 (2) = 8 ⇒ 2x = 4 ⇒ x = 2.

Answer: (x, y, z) = (2, 2, 2).

Ex 3 Find the inverse of the matrix $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ using elementary row operations.

Sol. Consider the augmented matrix formed by A and the identity matrix I.

$$[A|\ I] = \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

We will convert the right side matrix as the identity matrix.

Apply $R_3 \to 2R_3 + R_1$,

$$= \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 2 & 0 & 5 & 3 \end{bmatrix}$$

Now apply $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 5 & 2 & 0 & 0 & 8 \end{bmatrix}$$

Apply $R_1 \to 2R_1 - R_3$ and $R_2 \to 8R_2 - R_3$,

$$\begin{bmatrix} 1 & -3 & -2 & -4 & 0 & 0 \\ -1 & 3 & -2 & 0 & -8 & 0 \\ 1 & 5 & 2 & 0 & 0 & 8 \end{bmatrix}$$

Now divide R_1 by -4, R_2 by -8, and R_3 by 8:

$$\begin{bmatrix} -1/4 & +3/4 & +2/4 & 1 & 0 & 0 \\ +1/8 & -3/8 & 2/8 & 0 & 1 & 0 \\ 1/8 & 5/8 & 2/8 & 0 & 1 & 0 \end{bmatrix}$$

Now, the right side matrix got converted into I. Hence, the left side matrix is A^{-1}.

$$A^{-1} = \begin{bmatrix} -1/4 & 3/4 & 2/4 \\ 1/8 & -3/8 & 2/8 \\ 1/8 & 5/8 & 2/8 \end{bmatrix}$$