INVERSE TRIGONOMETRIC FUNCTIONS

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Properties Of Inverse Trigonometric Functions

Property 1 : "–x"

The graphs of $\sin^{-1}x$, $\tan^{-1} x$, $\mathrm{cosec}^{-1}x$ are symmetric about origin.

Hence we get $\sin^{-1}(-x) = -\sin^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}x$.

Also the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \pi/2)$. From this, we get

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$.

Property 2 : $T(T^{-1})$

(i) $\sin(\sin^{-1} x) = x, \quad -1 \le x \le 1$

Proof : Let $\theta = \sin^{-1}x$. Then $x \in [-1, 1]$ & $\theta \in [-\pi/2, \pi/2]$.

$\Rightarrow$ $\sin\theta = x$, by meaning of the symbol

$\Rightarrow$ $\sin(\sin^{-1} x) = x$

Similar proofs can be carried out to obtain

(ii) $\cos(\cos^{-1} x) = x, \quad -1 \le x \le 1$

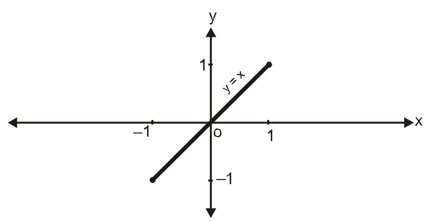
(iii) $\tan(\tan^{-1} x) = x, \quad x \in R$

(iv) $\cot(\cot^{-1} x) = x, \quad x \in R$

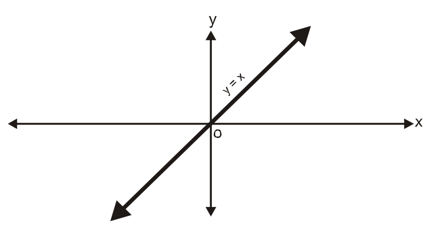
(v) $\sec(\sec^{-1} x) = x, \quad x \le -1, x \ge 1$

(vi) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x, \; |x| \ge 1$

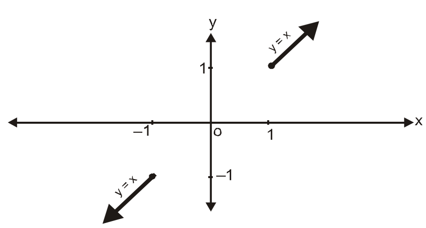
The graph of $y = \sin(\sin^{-1}x) \equiv \cos(\cos^{-1}x)$



The graph of $y = \tan(\tan^{-1}x) \equiv \cot(\cot^{-1}x)$



The graph of $y = \operatorname{cosec}(\operatorname{cosec}^{-1}x) \equiv \sec(\sec^{-1}x)$



Property 3 : $T^{-1}(T)$

(i) $\sin^{-1}(\sin x) = x$; $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos x) = x$; $0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $\text{cosec}^{-1}(\text{cosec}\, x) = x$; $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $x \neq 0$

(v) $\sec^{-1}(\sec x) = x$; $0 \le x \le \pi$ and $x \neq \frac{\pi}{2}$

(vii) $\cot^{-1}(\cot x) = x$; $0 < x < \pi$

Remark :

$\sin(\sin^{-1}x)$, $\cos(\cos^{-1}x)$, ... $\cot(\cot^{-1}x)$ are aperiodic (non periodic) functions where as $\sin^{-1}(\sin x)$, ..., $\cot^{-1}(\cot x)$ are periodic functions.

Ex.1: Find the value of $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$.

Sol: $\because$ $\tan^{-1}(\tan x) = x$

if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right) = \tan^{-1}\left(-\tan \frac{\pi}{4}\right)$$

$\tan^{-1}\left(-\tan \frac{\pi}{4}\right) = -\tan^{-1}\left(\tan \frac{\pi}{4}\right)$ (using property 1)

$\therefore -\tan^{-1}\left(\tan \frac{\pi}{4}\right) = -\frac{\pi}{4}$ (using property 3)

Ex.2: Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Sol: Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}(\sin 7) = \sin^{-1}\sin(7-2\pi)$

$\sin^{-1}\sin(7-2\pi) = 7-2\pi$ ($\because$ $7-2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$) (using property 3)

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

Let $y = \sin^{-1}(\sin -5)$

$\sin^{-1}(\sin -5) \neq -5$ as $-5 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}(\sin -5) = -\sin^{-1}\sin 5$ (using property 1)

$-\sin^{-1}\sin 5 = -\sin^{-1}\sin(5-2\pi)$

$-\sin^{-1}\sin(5-2\pi) = -(5-2\pi)$ ($\because$ $5-2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$)

Ex.3: Find the value of $\cos^{-1}\{\sin(-5)\}$

Sol: $\cos^{-1}\sin(-5) = \cos^{-1}(-\sin 5)$

$= \pi - \cos^{-1}(\sin 5)$ (using property 1)

$= \pi - \cos^{-1}\left(\cos\left(\frac{\pi}{2}-5\right)\right) = \pi - \cos^{-1}\left[\cos\left(\frac{5\pi}{2}-5\right)\right] = \pi - \left(\frac{5\pi}{2}-5\right)$ (using property 3)

$\pi - \left(\frac{5\pi}{2}-5\right) = 5 - \frac{3\pi}{2}$

Property 4 : "1/x"

(i) $\operatorname{cosec}^{-1}(x) = \sin^{-1}(1/x)$, $|x| \geq 1$

(ii) $\sec^{-1} x = \cos^{-1} (1/x)$, $|x| \geq 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

Property 5 : "π/2"

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, -1 \leq x \leq 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2},\ |x| \geq 1$

Ex.4: Find the value of cosec $\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\}$.

Sol: $\because$ $\cot(\cot^{-1} x) = x,\ \forall x \in R$

$\therefore$ $\cot\left(\cot^{-1}\frac{3\pi}{4}\right) = \frac{3\pi}{4}$ (using property 2)

$\operatorname{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\} = \operatorname{cosec}\left(\frac{3\pi}{4}\right) = \sqrt{2}.$

Ex.5: Find the value of tan $\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$

Sol: Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ....(i)

$\because$ $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$

(i) can be written as

$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\left(\cot^{-1}\frac{2}{3}\right) \quad \because \quad \cot^{-1} x = \tan^{-1}\frac{1}{x}$ if $x > 0$

$\therefore \quad y = -\tan\left(\tan^{-1}\frac{3}{2}\right) \quad \Rightarrow \quad y = -\frac{3}{2}$

Ex.6: Find the value of $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

Sol: $\sin\left(\tan^{-1}\frac{3}{4}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$

Ex.7: Find the value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

Sol: Let $y = \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$ ....(i)

Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta \quad \Rightarrow \quad \theta \in \left(0, \frac{\pi}{2}\right)$ and $\cos\theta = \frac{\sqrt{5}}{3}$

$\therefore$ (i) becomes $y = \tan\left(\frac{\theta}{2}\right)$ ....(ii)

$\because \quad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}} = \frac{3-\sqrt{5}}{3+\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$

$\tan\frac{\theta}{2} = \pm\left(\frac{3-\sqrt{5}}{2}\right)$ ....(iii)

$\frac{\theta}{2} \in \left(0, \frac{\pi}{4}\right) \quad \Rightarrow \quad \tan\frac{\theta}{2} > 0$

$\therefore$ from (iii), we get $y = \tan\frac{\theta}{2} = \left(\frac{3-\sqrt{5}}{2}\right)$

Property 6 : identities on addition and subtraction:

(i) $\sin^{-1}x + \sin^{-1}y \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\geq 0, y\geq 0 \ \& \ (x^2+y^2)\leq 1 \\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\geq 0, y\geq 0 \ \& \ x^2+y^2\geq 1 \end{cases}$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$; $x, y \in [0, 1]$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$; $x, y \in [0, 1]$

(iv) $\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right); & 0\leq x<y\leq 1 \\ -\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right); & 0\leq y<x\leq 1 \end{cases}$

(v) $\tan^{-1}x + \tan^{-1}y = \begin{cases} \pi/2 & \text{if } x, y>0 \ \& \ xy=1 \\ -\pi/2 & \text{if } x, y<0 \ \& \ xy=1 \\ \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x, y\geq 0 \ \& \ xy<1 \\ \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x, y\geq 0 \ \& \ xy>1 \end{cases}$

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, $x \geq 0, y \geq 0$

Notes :

(i) $x^2 + y^2 \leq 1 \ \& \ x, y \geq 0 \quad \Rightarrow \quad 0 \leq \sin^{-1}x + \sin^{-1}y \leq \frac{\pi}{2}$

and $x^2 + y^2 \geq 1 \ \& \ x, y \geq 0 \quad \Rightarrow \quad \frac{\pi}{2} \leq \sin^{-1}x + \sin^{-1}y \leq \pi$

(ii) $xy < 1$ and $x, y \geq 0 \Rightarrow 0 \leq \tan^{-1}x + \tan^{-1}y < \frac{\pi}{2}$; $xy > 1$ and $x, y \geq 0$

$\Rightarrow \frac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi$

(iii) For $x < 0$ or $y < 0$ these identities can be used with the help of property "$-x$"

i.e. change x or y to $-x$ or $-y$ which are positive .

INVERSE TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$; $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$; $0 \le x \le 1$

(iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$; $-1 < x < 1$

(iv) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$; $-\frac{1}{2} \le x \le \frac{1}{2}$

(v) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$; $0 \le x \le \frac{1}{2}$

(vi) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$; $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Ex.8 : Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\frac{84}{85}$

Sol: $\because$ $\frac{3}{5} > 0$, $\frac{15}{17} > 0$ and $\left(\frac{3}{5}\right)^2 + \left(\frac{15}{17}\right)^2 = \frac{8226}{7225} > 1$

$\therefore$ $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{225}{289}} + \frac{15}{17}\sqrt{1-\frac{9}{25}}\right)$

$= \pi - \sin^{-1}\left(\frac{3}{5}\cdot\frac{8}{17} + \frac{15}{17}\cdot\frac{4}{5}\right) = \pi - \sin^{-1}\left(\frac{84}{85}\right)$

Ex.9 : Evaluate $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

Sol: Let $z = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{4}{5} - \tan^{-1}\frac{63}{16}$

$\because$ $\sin^{-1}\frac{4}{5} = \frac{\pi}{2} - \cos^{-1}\frac{4}{5}$

$\therefore$ $z = \cos^{-1}\frac{12}{13} + \left(\frac{\pi}{2} - \cos^{-1}\frac{4}{5}\right) - \tan^{-1}\frac{63}{16}.$

$z = \frac{\pi}{2} - \left(\cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13}\right) - \tan^{-1}\frac{63}{16}$ ....(i)

$\because$ $\frac{4}{5} > 0,\ \frac{12}{13} > 0$ and $\frac{4}{5} < \frac{12}{13}$

$\therefore$ $\cos^{-1}\frac{4}{5} - \cos^{-1}\frac{12}{13} = \cos^{-1}\left[\frac{4}{5}\times\frac{12}{13} + \sqrt{1-\frac{16}{25}}\ \sqrt{1-\frac{144}{169}}\right] = \cos^{-1}\left(\frac{63}{65}\right)$

$\therefore$ equation (i) can be written as

$z = \frac{\pi}{2} - \cos^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right)$

$z = \sin^{-1}\left(\frac{63}{65}\right) - \tan^{-1}\left(\frac{63}{16}\right)$ ....(ii)

$\because$ $\sin^{-1}\left(\frac{63}{65}\right) = \tan^{-1}\left(\frac{63}{16}\right)$

$\therefore$ from equation (ii), we get

$\therefore$ $z = \tan^{-1}\left(\frac{63}{16}\right) - \tan^{-1}\left(\frac{63}{16}\right)$ $\Rightarrow$ $z = 0$

Simplification :

Ex.10: Define $y = \cos^{-1}(4x^3 - 3x)$ in terms of $\cos^{-1} x$ when x $\left[\frac{1}{2}, 1\right]$

Sol: Let $y = \cos^{-1}(4x^3 - 3x)$

$\because$ Domain : $[-1, 1]$ and range : $[0, \pi]$

Let $\cos^{-1} x = \theta \quad \Rightarrow \quad \theta \in [0, \pi]$ and $x = \cos\theta$

$\therefore \quad y = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$y = \cos^{-1}(\cos 3\theta)$

$\because \quad x \in \left[\frac{1}{2}, 1\right] \quad \therefore \theta \in \left[0, \frac{\pi}{3}\right]$

$\therefore 3\theta \in [0,\pi]$

$\therefore \cos^{-1}(\cos 3\theta) = 3\theta = 3(\cos^{-1} x)$

Ex.11: Define $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ in terms of $\sin^{-1} x$ when $|x| \leq \frac{1}{\sqrt{2}}$

Sol: Put $x = \sin\theta$; $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \Rightarrow 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$ (using property 3)

$\therefore 2\theta = 2(\sin^{-1} x)$

Ex.12: Define $\cos^{-1}(2x^2 - 1)$ in terms of $\cos^{-1} x$ when $0 \leq x \leq 1$

Sol: Put $x = \cos\theta$; $\theta \in \left[0, \frac{\pi}{2}\right] \Rightarrow 2\theta \in [0,\pi]$

$\therefore \cos^{-1}(2x^2 - 1) = \cos^{-1}(\cos 2\theta) = 2\theta$ (using property 3)

$\therefore 2\theta = 2(\cos^{-1} x)$