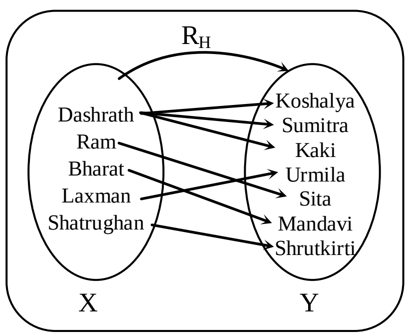
RELATIONS AND FUNCTIONS

TYPES OF RELATION

Relations :

A relation R from set X to Y ($R : X \rightarrow Y$) is a correspondence between set X to set Y by which some or more elements of X are associated with some or more elements of Y. Therefore a relation (or binary relation) R, from a non-empty set X to another non-empty set Y, is a subset of $X \times Y$. i.e. $R_H : X \rightarrow Y$ is nothing but subset of $A \times B$.

e.g. Consider a set X and Y as set of all males and females members of a royal family of the kingdom Ayodhya X = {Dashrath, Ram, Bharat, Laxman, shatrughan} and Y = {Koshaliya, Kakai, sumitra, Sita, Mandavi, Urmila, Shrutkirti} and a relation R is defined as "was husband of "from set X to set Y.



Then R_H = {(Dashrath, Koshaliya), (Ram, sita), (Bharat, Mandavi), (Laxman, Urmila), (Shatrughan, Shrutkirti), (Dashrath, Kakai), (Dashrath, Sumitra)}

Note :

(i) If a is related to b then symbolically it is written as a R b where a is pre-image and b is image

(ii) If a is not related to b then symbolically it is written as $a \not R b$.

Domain, Co-domain & Range of Relation :

Domain :

of relation is collection of elements of the first set which are participating in the correspondence i.e. it is set of all pre-images under the relation R. e.g. Domain of R_H : {Dashrath, Ram, Bharat, Laxman, Shatrughan}

Co-Domain :

All elements of set Y irrespective of whether they are related with any element of X or not constitute co-domain. e.g. Y = {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti} is co-domain of R_H.

Range :

of relation is a set of those elements of set Y which are participating in correspondence i.e. set of all images. Range of R_H : {Koshaliya, Kakai, Sumitra, Sita, Mandavi, Urmila, Shrutkirti}.

EX.1 Let A = {1, 3, 4, 5, 7} and B = {1, 4, 6, 7} and R be the relation 'is one less than' from A to B, then list the domain, range and co-domain sets of R.

Sol: R = {(3, 4), (5, 6)}

So, Dom(R) = {3, 5}

Range of R = {4, 6}

Co-domain of R = B = {1, 4, 6, 7}

Clearly Range of R $\subseteq$ co-domain of R.

Inverse Relation

Let $R \subseteq A \times B$ be a relation from A to B.

The inverse relation of R (denoted by R^{-1}) is a relation from B to A defined as $R^{-1} = \{(b, a) : (a, b) \in R\}$

If $(a, b) \in R$, then $(b, a) \in R^{-1}$, " $a \in A$, $b \in B$.

domain of R^{-1} = Range of R

Range of R^{-1} = domain of R

$(R^{-1})^{-1} = R$

Identity Relation

The identity relation on a set A is the set of ordered pairs belonging to $A \times A$ and is denoted by I_A.

$I_A = \{(a, a) : a \in A\}$

i.e. every element of A is related to only itself.

R is an identity relation if $(a, b) \in R$ iff $a = b$, $a \in A$, $b \in A$.

$\mathbf{I_A^{-1} = I_A}$

Domain of I_A= Range of I_A = A

Ex.2: 'is equal to' is an identity relation on set of Natural number (N)

i.e. $\{(1, 1), (2, 2), (3, 3)\ldots\ldots\} = I_N$

Universal Relation

If A be a set and R is the set $A \times A$, then R is called universal relation in A.

If $R = A \times A$, then R is universal relation in A.

Void Relation

f is called the empty or void relation if $f \subset A \times A$

Types of Relations on a Set

If A is a non-empty set, then a relation R on A is said to be

1. Reflexive :

If $(a, a) \in R$, " $a \in A$.

i.e. a R a, " a Î A

“is equal to”, “is a friend of”, “is parallel to”, are some of reflexive relations.

2. Symmetric :

If a R b $\Rightarrow$ b R a, " a, b $\in$ A

i.e. if (a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R, " a, b $\in$ A

"is a friend of", "is parallel to", "is equal to", are some of symmetric relations.

3. Anti-Symmetric :

If a R b and b R a $\Rightarrow$ a = b, " a, b $\in$ A (If $R \cap R^{-1}$ = Identity, then R is anti-symmetric)

"is divisible by" is an anti symmetric relation.

4. Transitive :

If a R b and b R c $\Rightarrow$ a R c, " a, b, c $\in$ A

i.e. If (a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ (a, c) Î R, " a, b, c $\in$ A

"is parallel to", "is equal to", "is congruent to" are some of the transitive relation.

Equivalence Relation

A relation R on a non-empty set A is called an equivalence relation if and only if it is Reflexive, Symmetric as well as Transitive.

"is parallel to", "is equal to", "is congruent to" "Identity relation" are some of the equivalence relations.

Every identity relation is an equivalence relation but every equivalence relation need not to be identity relation.

Ex.3: Check the following relations for being reflexive, symmetric, transitive and thus choose the equivalence relations if any.

(i) a R b if $\leq$ b; a, b $\in$ set of real numbers.

(ii) a R b iff a $<$ b; a, b $\in$ N.

(iii) a R b iff $>$; a, b $\in$ R.

(iv) a R b iff a divides b; a, b $\in$ N.

(v) a R b iff (a – b) is divisble by n; a, b $\in$ I, n is a fixed positive integer.

Sol:

(i) Not reflexive, not symmetric but transitive

Let $a = -2$ and $b = 3$; $(-2, 3) \in R$. Since ≤ 3 is true

Since $= 2$ -2 hence relation is not Reflexive

Since ≤ -2 is wrong hence relation is not symmetric

Now Let a, b, c be three real numbers such that $\le b$ and $\le c$

$\le b \Rightarrow b\ ^{3}\ 0$, so $\le c$ $\Rightarrow b \le c$

Hence $\le c$ is true so the given relation is transitive.

(ii) Not reflexive, not symmetric but transitive.

Since no natural number is less than itself hence not reflexive,

If $a < b$ then $b < a$ is false. Hence not symmetric.

If $a < b$ then $b < c$ clearly $a < c$. Hence transitive

(iii) Not reflexive, symmetric, not transitive.

hence it is not reflexive.

hence symmetric.

Let $a = 1$, $b = -1$ and $c =$, so $(b, c) \in R$

But . Hence R is not a transitive relation.

(iv) Reflexive, not symmetric, transitive

Since $= 1$ i.e. every number divides itself, hence R is reflexive.

If a divides b then b does not divide a (unless $(a = b)$ hence the relation is not symmetric (but anti-symmetric).

If a divides b and b divides c then it is clear that a will divide c. Hence transitive.

(v) Relfexive, symmetric as well as transitive, hence it is an equivalence relation.

Since 0 is divisible by n so given relation is reflexive

If $a - b$ is divisible by n, then $(b - a)$ will also be divisible by n. Hence, symmetric.

If $a - b = nI_1$ and $b - c = nI_2$, where I_1, I_2 are integer.

Then, $a - c = (a - b) + (b - c) = n(I_1 + I_2)$ so $a - c$ is also divisible by n, hence transitive.

Ordered Relation

R is an ordered relation if it is transitive but not equivalence relation. e.g. a R b iff $a < b$, a, b $\in$ N is an ordered relation.

e.g. R = {(1, 1), (1, 3), (1, 2), (2, 1), (2, 2) (2, 3)} is not reflexive, not symmetric and transitive, hence not an equivalence relation.

so, R is an ordered relation.

Partial Order Relation

R is an partial order relation if it is reflexive, transitive and antisymmetric at the same time.

Ex.4 : a R b iff a divides b; a, b $\in$ N is partial order relation since it is reflexive, transitive and anti-symmetric.

If R is reflexive $\Rightarrow R^{-1}$ is reflexive

If R is symmetric $\Rightarrow R^{-1}$ is symmetric

If R is transitive $\Rightarrow R^{-1}$ is transitive

Hence if R is an equivalence relation $\Rightarrow R^{-1}$ is equivalence relation

COMPOSITION OF TWO RELATIONS

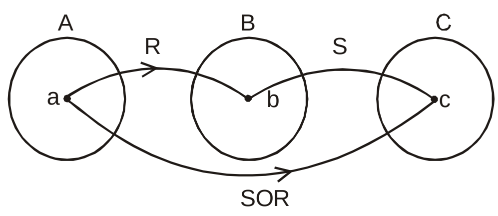
If A, B, C are three sets such that $R \subseteq A \times B$ and $S \subseteq B \times C$ then $SOR \subseteq A \times C$.

SOR, ROS, $(SOR)^{-1}$, $S^{-1}OR^{-1}$, $S^{-1}OR$ etc. are called compositions of two relations.

$(SOR)^{-1} = R^{-1}OS^{-1}$

$(R_1OR_2OR_3.....OR_n)^{-1} = R_n^{-1}OR_{n-1}^{-1}\ O......OR_3^{-1}OR_2^{-1}OR_1^{-1}$

Pictorially



Ex.5 Let R be a relation such that R = {(a, d), (c, g) (d, e), (d, f) (g, f)} then find

(i) $R^{-1}OR^{-1}$ (ii) $(ROR^{-1})^{-1}$

Sol: $R^{-1}OR^{-1} = (ROR)^{-1}$ and $(ROR^{-1})^{-1} = ROR^{-1}$

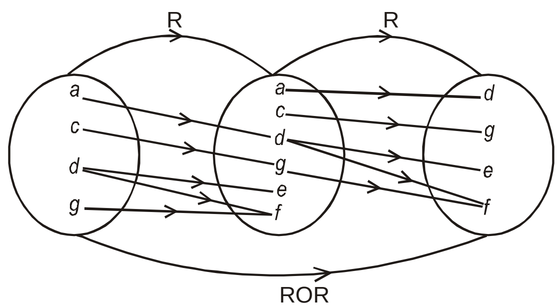
Clearly R^{-1} = {(d, a), (g, c), (e, d), (f, d), (f, g)}

Domain of R = {a, c, d, g}; Range of R = {d, g, e, f}

Domain of R^{-1} = {d, g, e, f}; Range of R^{-1} = {a, c, d, f}

(i) R = {(a, d), (c, g) (d, e), (d, f) (g, f)}

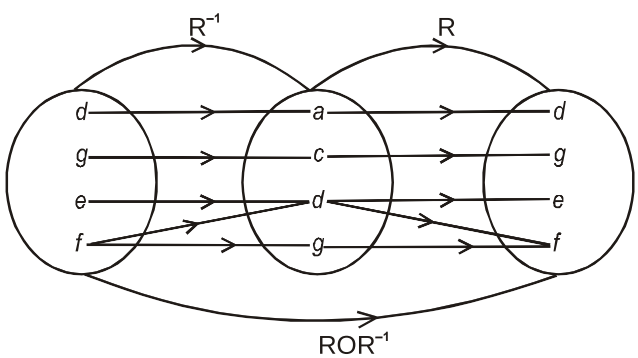
R^{-1}={(d, a), (g, c), (e, d), (f, d), (f, g)}



From above figure clearly ROR = {(a, e), (a, f), (c, f)}

so, $R^{-1}\,OR^{-1} = (ROR)^{-1}$ = {(*e*, *a*), (*f*, *a*), (*f*, *c*)}

(ii)



Hence ROR^{-1} = {(d, d), (g, g), (e, e), (e, f), (f, e), (f, f)}

$(ROR^{-1})^{-1} = ROR^{-1}$ = {(d, d), (g, g), (e, e), (e, f), (f, e), (f, f)}