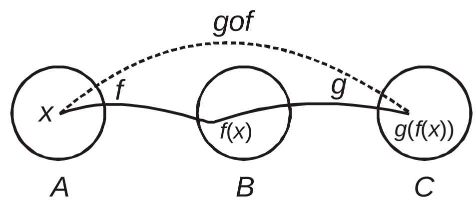
Relations and Functions

Composite and Invertible Function

Composition of Function



Let f : A$\to$ B and g : B $\to$ C then the composition of g and f is denoted by gof and is defined as gof : A$\to$ C given by gof (x) = g(f(x))

Similarly fog is defined. Note that, gof is defined only if Range f $\subseteq$ dom g and fog is defined only if Range $\subseteq$ dom f. dom fog = {x $\in$ dom g : g(x) $\in$ dom f}

Note :

(a) Function gof will exist only when range of f is the subset of domain of g.

(b) gof (x) is simply the g-image of f(x), where f(x) is f-image of elements x $\in$ A.

(c) fog does not exist here because range of g is not a subset of domain of f.

Properties of composite function :

(a) If f and g are two functions then for composite of two functions fog $\neq$ gof .

(b) Composite functions obeys the property of associativity i.e. fo (goh) = (fog) oh.

(c) Composite function of two one-one onto functions if exist, will also be a one-one onto function.

Ex.1 Let f(x) = $x^2 + 3$ and g(x) = $\sqrt{x}$. Since dom g = $[0, \infty)$, dom f = R

Sol. we have fog (x) = f(g(x)) = f($\sqrt{x}$)= $(\sqrt{x})^2$+3=x+3

So dom fog = $\{x \in [0, \infty) : g(x) \in R\} = [0, \infty)$

Let us now find gof, we have (gof) (x) = g(f(x)) = g(x^2 +3) = $\sqrt{x^2+3}$,

then dom gof = $\{x \in R : f(x) \in [0, \infty)\}$ = R.

Ex.2 : Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Sol: (i) Domain of f is $[-3, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[1, \infty)$.

For gof(x)

Since range of f is a subset of domain of g,

domain of gof is $[-3, \infty)$ {equal to the domain of f }

gof (x) = g{f(x)} = g () = 1 + $\sqrt{x+3}$ (x+3) = x + 4. Range of gof is $[1, \infty)$.

For fog(x)

since range of g is a subset of domain of f,

$\therefore$ domain of fog is R {equal to the domain of g}

fog (x) = f{g(x)}= f(1+ x^2) = $\sqrt{x^2+4}$ Range of fog is $[2, \infty)$.

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is $[0, \infty)$, range of f is $[0, \infty)$.

Domain of g is R, range of g is $[-1, \infty)$.

For gof(x)

Since range of f is a subset of the domain of g,

domain of gof is $[0, \infty)$ and $g\{f(x)\}= g(\sqrt{x}) = x - 1$. Range of gof is $[-1, \infty)$

For fog(x)

Since range of g is not a subset of the domain of f i.e. $[-1, \infty) \not\subset [0, \infty)$

$\therefore$ fog is not defined on whole of the domain of g.

Domain of fog is $\{x \in R$, the domain of $g : g(x) \in [0, \infty)$, the domain of $f\}$.

Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$

i.e. $D = \{ x \in R: 0 \le x^2 - 1\} = \{ x \in R: x \le -1 \text{ or } x \ge 1 \} = (-\infty, -1] \cup [1, \infty)$

fog $(x) = f\{g(x)\} = f(x^2-1) = \sqrt{x^2-1}$ Its range is $[0, \infty)$.

Ex.3 : Let $f(x) = \cos x + x$ and $g(x) = x^2$. Find fog(x)

Sol: $fog(x) = \cos g(x) + g(x)$

$= \cos x^2 + x^2$

Ex.4 : If $f(x) = | |x - 3| - 2 |$ $\quad 0 \le x \le 4$

$g(x) = 4 - |2 - x|$ $\quad -1 \le x \le 3$

then find fog(2)

Sol: $fog(2) = f(4)$ $(\because g(2) = 4)$

$\therefore fog(2) = 1$

Inverse Function

Two functions f and g are inverse of each other if $f(g(x)) = x$ for $x \in$ dom g and $g(f(x)) = x$ for $x \in$ dom f ,

i.e., gof $= I_{dom}$ f and fog $= I_{dom}$ g where I_{dom} f is identity function on dom f and I_{dom} g is identity function on dom g. We denote g by f^{-1} or f by g^{-1}. To find the inverse of f, write down the equation $y = f(x)$ and then solve x as a function of y. The resulting equation is $x = f^{-1}(y)$.

Note : For the existence of inverse function, it should be one-one and onto.

Properties :

(a) Inverse of a bijection is also a bijection function.

(b) Inverse of a bijection is unique.

(c) $(f^{-1})^{-1} = f$

(d) If f and g are two bijections such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$

(e) If $f : A \to B$ is a bijection then $f^{-1} : B \to A$ is an inverse function of f .

$f^{-1}of = I_A$ and $fof^{-1} = I_B$.

Here I_A, is an identity function on set A, and I_B, is an identity function on set B.

Ex.5: To find the inverse of $f(x) = \dfrac{e^x - e^{-x}}{2}$

Sol. We write $y = \dfrac{e^x - e^{-x}}{2} \Rightarrow 2y = \dfrac{e^{2x} - 1}{e^x}$

$$e^{2x} - 2ye^x - 1 = 0 \Rightarrow e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$

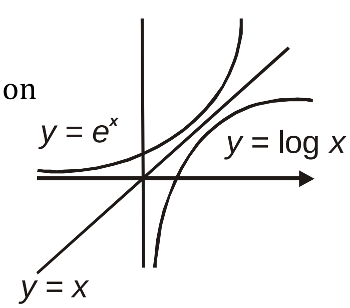
$e^x = y \pm \sqrt{y^2 + 1}$ since $e^{x\,3}$ 0 so $e^x = y + \sqrt{y^2 + 1}$

$x = \log\left(y + \sqrt{y^2 + 1}\right)$ $x = \log\left(y + \sqrt{y^2 + 1}\right)$

Thus $f^{-1}(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

The graph of f and f^{-1} are related to each other in the following way :

If the point (x, y) lies on the graph of f then the point (y, x) lies on the graph of f^{-1} and vice versa. Thus the graph of f^{-1} is the reflection of the graph of f in the line y = x as below (since we know that y = log x and y = e^x are inverse of each other).



Existence of inverse function

A function need not have an inverse. e.g. the function $f(x) = x^2$ has no inverse (where dom f = R). To have an inverse, a function must be both one-one and onto, i.e. bijective.

Ex. 6 : (i) Determine whether $f(x) = \frac{2x+3}{4}$ for $f : R \to R$, is bijective or not ? If so find it $f^{-1}(x)$

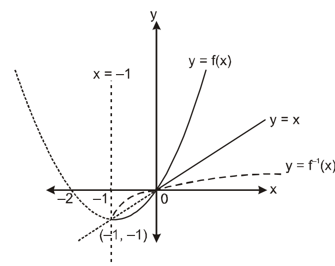
(ii) Let $f(x) = x^2 + 2x;\ x \geq -1$. Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$

(iii) If $y=f(x)=x^2 - 3x + 1,\ x \geq 2$. Find the value of $g'(1)$ where g is inverse of f

Sol: (i) Given function is one-one and onto, therefore it is invertible.

$$y = \frac{2x+3}{4} \Rightarrow x = \frac{4y-3}{2} \quad \therefore \quad f^{-1}(x) = \frac{4x-3}{2}$$

(ii) $f(x) = f^{-1}(x)$ is equivalent to $f(x) = x$



$$\Rightarrow x^2 + 2x = x \qquad \Rightarrow x(x + 1) = 0 \qquad \Rightarrow x = 0, -1$$

Hence two solution for $f(x) = f^{-1}(x)$

(iii) $y = 1 \qquad \Rightarrow x^2 - 3x + 1 = 1$

$\Rightarrow x(x - 3) = 0 \qquad \Rightarrow x = 0, 3$

But $x \geq 2 \qquad \therefore \qquad x = 3$

Now $g(f(x)) = x$

Differentiating both sides w.r.t. x

$$\Rightarrow g'(f(x)).\ f'(x) = 1 \quad \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

$$\Rightarrow g'(f(3)) = \frac{1}{f'(3)} \qquad \Rightarrow g'(1) = \frac{1}{6-3} = \frac{1}{3} = (\text{As } f'(x) = 2x - 3)$$

Alternate Method

$y = x^2 - 3x + 1$

$x^2 - 3x + 1 - y = 0$

$x = \dfrac{3 \pm \sqrt{9-4(1-y)}}{2} = \dfrac{3 \pm \sqrt{5+4y}}{2}$

$x \geq 2$

$x = \dfrac{3+\sqrt{5+4y}}{2}$

$g(x) = \dfrac{3+\sqrt{5+4x}}{2}$

$g'(x) = 0 + \dfrac{1}{4\sqrt{5+4x}}4$

$g'(1) = \dfrac{1}{\sqrt{5+4}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

DOMAIN AND RANGE OF SOME STANDARD FUNCTION

Function	**Domain**	**Range**
Polynomial function	R	R
Identity function x	R	R
Constant function c	R	{c}
Reciprocal fn 1/x	R_0	R_0
Signum function	R	{–1,0,1}
$ax + b$; $a, b \in R$	R	R
$ax^3 + b$; $a, b \in R$	R	R
x^2, $\lvert x\rvert$	R	$R^+ \cup \{0\}$
x^3, $x\lvert x\rvert$	R	R

$x + \|x\|$	R	$R^+ \cup \{0\}$
$x - \|x\|$	R	$R^- \cup \{0\}$
$[x]$	R	Z
$x - [x]$	R	$[0,1)$
$\|x\|/x$	R_0	$\{-1,1\}$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
a^x	R	R^+
$\log x$	R^+	R
$\sin x$	R	$[-1,1]$
$\cos x$	R	$[-1,1]$
$\tan x$	$R - \{(2n+1)\pi/2 \mid n \in z\}$	R
$\cot x$	$R - \{n\pi \mid n \in z\}$	R
$\sec x$	$R - \{(2n+1)\pi/2 \mid n \in z\}$	$R - (-1,1)$
$\text{cosec } x$	$R - \{n\pi \mid n \in z\}$	$R - (-1,1)$
$\sin^{-1} x$	$[-1,1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1,1]$	$[0, \pi]$
$\tan^{-1} x$	R	$(-\pi/2, +\pi/2)$
$\cot^{-1} x$	R	$(0,\pi)$
$\sec^{-1} x$	$R - (-1,1)$	$[0,\pi] - \{\pi/2\}$
$\text{cosec}^{-1} x$	$R - (-1,1)$	$(-\pi/2,\pi/2]-\{0\}$

Ex.7 Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Sol: $\tan^{-1}(\tan x) = x$

if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right) = \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$

$\tan^{-1}\left(-\tan\frac{\pi}{4}\right) = -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$ (using property 1)

$\therefore\ -\tan^{-1}\left(\tan\frac{\pi}{4}\right) = -\frac{\pi}{4}$ (using property 3)