VECTORS

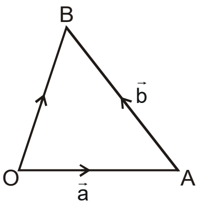
1.1 Basic concept

- Vectors have magnitude and direction, whereas scalars have only magnitude.
- Vector. A directed line segment represents a vector. A vector from P to Q is denoted by $\vec{PQ}$. P and Q are called respectively initial and terminal points of the vector $\vec{PQ}$. The direction of vector $\vec{PQ}$ is from P to Q.
- The magnitude of vector $\vec{PQ}$ is denoted by $|\vec{PQ}|$ and represents length of line segment PQ.
- Zero vector. A vector whose initial and terminal points coincide is called a zero vector. It is also defined as a vector whose magnitude is zero. It is denoted by $\vec{0}$ or $\vec{AA}, \vec{BB}$ etc. $|\vec{0}| = 0$, $|\vec{0}| = 0$, $|\vec{AA}| = 0$.
- Unit vector. A vector $\vec{a}$ is called a unit vector if its magnitude is one unit. It is denoted by $\hat{a}$, $|\hat{a}| = 1$. Unit vector represents direction along a vector $\vec{a}$, also $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.
- **Position vector of a point.** If we take a fixed point O, which is called the origin of reference and P be any point in the plane, then the vector $\vec{OP}$ is called the position vector of P relative to O. If position vector of point P is $\vec{a}$, denoted as P($\vec{a}$), then corresponding to some origin of reference O, $\vec{OP} = \vec{a}$.
- **Line of support.** A line, whose segment is PQ, is called the line of support of the vector $\vec{PQ}$.

1.2 Types of vectors

1. **Like vectors.** Two vectors are said to be like vectors, if they have (i) same or parallel lines of support.
   (ii) same direction.
2. **Unlike vectors.** Two vectors are said to be unlike vectors, if (i) their lines of support are same or parallel, (ii) opposite direction.
3. **Equal vectors.** Two vectors are said to be equal, if they have (i) same or parallel lines of support, (ii) same direction and (iii) equal magnitudes.
   We can also say that if like vectors have equal magnitudes then they are equal.
4. **Coinitial vectors.** Two or more vectors are said to be coinitial vectors if they have the same initial point, e.g., $\vec{AP}, \vec{AQ}, \vec{AR}$ etc. are coinitial vectors.
5. **Collinear vectors.** Two or more vectors are called collinear vectors, if they have the same or parallel lines of support.
6. **Coplanar vectors.** Any number of non-zero vectors are said to be coplanar if they lie in the same plane or parallel planes.
7. **Negative vector.** A vector which has the same magnitude but opposite direction of the given vector, is called the negative of the given vector. If $\vec{PR} = \vec{x}$ then $\vec{RP} = -\vec{x}$.

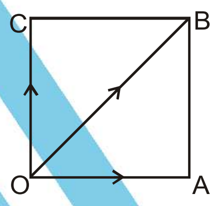
1.3 Properties with respect to addition



- Triangle law of vector addition. If two vectors are represented along two sides of a triangle taken in order, then their resultant is represented by the third side taken in the opposite order.

  If the sides OA and AB of ΔOAB represent $\overrightarrow{OA}$ and $\overrightarrow{AB}$,

  then $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, i.e., $\overrightarrow{OB} = \vec{a} + \vec{b}$



- **Parallelogram law of vector addition.** If two vectors are represented along two adjacent sides of a parallelogram, then their resultant is represented along the diagonal of a parallelogram passing through the common vertex of adjacent sides. If the sides OA and OC of a parallelogram OABC represent respectively
  $\overrightarrow{OA}$ and $\overrightarrow{OC}$, then $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$.

- **Properties with respect to addition**

  (i) **Commutative.** For vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

  (ii) **Associative.** For vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

  (iii) **Additive identity**. For every vector $\vec{a}$, a zero vector $\vec{0}$ is its additive identity as $\vec{a} + \vec{0} = \vec{a}$.

  (iv) **Additive inverse.** For a vector $\vec{a}$, a negative of vector $\vec{a}$ is its additive inverse as $\vec{a} + (-\vec{a}) = \vec{0}$.

  Also for vector $\overrightarrow{AB}$, its additive inverse is $\overrightarrow{BA}$ as $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \vec{0}$, Also, we conclude $\overrightarrow{BA} = -\overrightarrow{AB}$.

1.4 Properties of with respect to multiplication

- **Multiplication of a vector by a scalar.** Let $\vec{a}$ be a given vector and k a scalar, then multiplication of vector $\vec{a}$ by scalar k, denoted k $\vec{a}$, is a vector whose magnitude is k times that of vector $\vec{a}$ and direction is (i) same as that of $\vec{a}$, if k > 0. (ii) opposite to that of $\vec{a}$, if k < 0. (iii) a zero vector, if k = 0.
- To prove $\vec{a}$ is parallel to $\vec{b}$ we have to show that $\vec{a} = k\,\vec{b}$, where k is a scalar.
- Properties with respect to multiplication of a vector by a scalar

  For vectors $\vec{a}$, $\vec{b}$ and scalars ℓ, m, we have

  (i) $\ell(\vec{a} + \vec{b}) = \ell\vec{a} + \ell\vec{b}$ (ii) $(\ell + m)\vec{a} = \ell\vec{a} + m\vec{a}$ (iii) $\ell(m\vec{a}) = (\ell m)\vec{a}$.

1.5 Propertes of vectors

- To find a vector when position vectors of end points are given :

  Let $\vec{a}$ and $\vec{b}$ be the position vectors of end points A and B of a line segment AB. Then, $\overrightarrow{AB}$ = Position vector of B – Position vector of A = $\overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$.

- **Components of a vector.** If $\vec{r} = \vec{a} + \vec{b}$, then $\vec{a}$ and $\vec{b}$ are known as components of $\vec{r}$.
- **Vector in two dimensions :**

  Let $\vec{r}$ be position vector of a point P(x, y),

(i) Then $\vec{r} = \overrightarrow{OP} = x\hat{i} + y\hat{j}$

(ii) $x\hat{i}$ and $y\hat{j}$ are known as component vectors of $\vec{r}$ along x and y-axis and x and y are known as components of $\vec{r}$ along x and y-axis.

(iii) Magnitude of $\vec{r}$, is given by, $|\vec{r}| = \sqrt{x^2 + y^2}$.

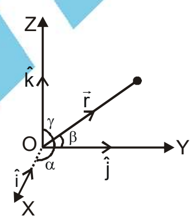
(iv) Unit vector along $\vec{r}$, is given by, $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j}}{|\vec{r}|}$, i.e., $\hat{r} = \frac{x}{|\vec{r}|}\hat{i} + \frac{y}{|\vec{r}|}\hat{j}$.

(v) If θ is angle which $\vec{r}$ makes with x-axis, then $\cos\theta = \frac{x}{|\vec{r}|}$, $\sin\theta = \frac{y}{|\vec{r}|}$.

(vi) $\cos\theta$, $\sin\theta$ are known as direction cosines of $\vec{r}$.

(vii) For $\vec{r} = x\hat{i} + y\hat{j}$; x and y are known as direction ratios of $\vec{r}$, i.e., components of a vector are direction ratios of $\vec{r}$, but converse may or may not be true.

- **Vector in three dimensions :**
  If $\vec{r}$ is position vector of point P(x, y, z)



(i) Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are component vectors and x, y and z are components of vector $\vec{r}$ along x, y and z-axis.

(ii) **Magnitude** of $\vec{r}$, is given by, $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

(iii) If α, β, γ are the angles which vector $\vec{r}$ makes with x, y and z-axis, then its direction consies

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are $\cos\alpha = \frac{x}{|\vec{r}|}$; $\cos\beta = \frac{y}{|\vec{r}|}$; $\cos\gamma = \frac{z}{|\vec{r}|}$.

(iv) **Unit vector** along $\vec{r}$, is given by, $\hat{r} = \frac{x}{|\vec{r}|}\hat{i} + \frac{y}{|\vec{r}|}\hat{j} + \frac{z}{|\vec{r}|}\hat{k} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$

(v) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then x, y, z are direction ratios of $\vec{r}$, i.e., components of a vector are direction ratios of a vector.

- **Vector when coordinates of end points are given.** If A (x_1, y_1, z_1) and B(x_2, y_2, z_2) be the end-points of line segment AB, then $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

  $(x_2 - x_1)$, $(y_2 - y_1)$ and $(z_2 - z_1)$ are components of $\overrightarrow{AB}$ along x, y and z-axis respectively. These are also direction ratios along x, y and z-axis respectively.
- Position vector of a point dividing the line segment in a given ratio (**Section formula**).
  Position vector $\vec{r}$ of a point which divides the join of two given points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio $\ell : m$ internally is, $\vec{r} = \frac{\ell\vec{b} + m\vec{a}}{\ell + m}$.

  If point divides externally, in the ratio l : m, then position vector is $\frac{\ell\vec{b} - m\vec{a}}{\ell - m}$.
- Position vector of the mid-point (1 : 1) of the line segment joining the end points with position vectors $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a} + \vec{b}}{2}$.

1.6 Scalar or dot product of two vectors

- Scalar or dot product of two vectors. If θ is the angle between two given vectors $\vec{a}$ and $\vec{b}$, then their scalar or dot product, denoted by $\vec{a} \cdot \vec{b}$ is given by $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$, where θ is angle between $\vec{a}$ and $\vec{b}$.

- $\vec{a}\cdot\vec{b}$ is a scalar quantity.
- $(\vec{a}\cdot\vec{b})\cdot\vec{c}$ is not defined.
- **Commutative property.** For vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.
- **Associative property.** For vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, associative property does not hold as $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ is not defined.
- **Distributive property.** For vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, ; $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.
- For vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot(\lambda\vec{b}) = (\lambda\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b})$, λ is a scalar.
- For vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{0}$, $\vec{b} = \vec{0}$ or $\vec{a}\perp\vec{b}$.
- For vector $\vec{a}$, $\vec{a}\cdot\vec{a} = \vec{a}^2 = |\vec{a}|^2$, i.e., square of a vector is equal to square of its magnitude.
- Projection of $\vec{a}$ along $\vec{b}$ is $\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$ and projection vector of $\vec{a}$ along $\vec{b}$ is $\left(\dfrac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$.
- Angle between $\vec{a}$ and $\vec{b}$ is given by, $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$, θ is the angle between $\vec{a}$ and $\vec{b}$.
- If $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along x, y and z-axis respectively, then
  $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$; $\hat{i}\cdot\hat{j} = 0$, $\hat{j}\cdot\hat{k} = 0$, $\hat{k}\cdot\hat{i} = 0$.
- If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1a_2 + b_1b_2 + c_1c_2$ If $\vec{a}\perp\vec{b}$ then $\vec{a}\cdot\vec{b} = 0 \Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If θ is angle between $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

1.7 Vector or cross product of two vectors

- Vector or cross product of two vectors $\vec{a}$ and $\vec{b}$. denoted by $\vec{a}\cdot\vec{b}$, is given by
  $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,n$, θ is angle between $\vec{a}$ and $\vec{b}$, $\hat{n}$ is a unit vector $\perp$ to $\vec{a}\times\vec{b}$, is given by
  $\vec{a}$ and $\vec{b}$ and direction is such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right hand system.
- $\vec{a}\times\vec{b}$ is a vector quantity, whose magnitude is, $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$.
- If θ is angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$.
- For $\vec{a}$, $\vec{a}\times\vec{a} = \vec{0}$.
- For vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$, i.e., cross product of two vectors is not commutative.
- For $\vec{a}$, $\vec{b}$ and $\vec{c}$, $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$, in general. Not associative.
- **Distributive property**. For vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$.
- For vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a} = \vec{0}$, $\vec{b} = \vec{0}$ or $\vec{a}\,||\,\vec{b}$.
- If we want to show that two non-zero vectors $\vec{a}$ and $\vec{b}$ are parallel, then we should show that $\vec{a}\times\vec{b} = \vec{0}$.
- Geometrically, $|\vec{a}\times\vec{b}|$ represents area of a parallelogram whose adjacent sides are along $\vec{a}$ and $\vec{b}$.
- Area of a triangle whose sides are along $\vec{a}$ and $\vec{b}$ is given by $\frac{1}{2}|\vec{a}\times\vec{b}|$.
- Area of a parallelogram whose diagonals are along $\vec{d}_1$ and $\vec{d}_2$ is given by $\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$.
- If $\hat{i}$, $\hat{j}$, $\hat{k}$ are vectors along x, y and z-axis respectively, then
  $\hat{i}\times\hat{i} = \vec{0}$, $\hat{j}\times\hat{j} = \vec{0}$, $\hat{k}\times\hat{k} = \vec{0}$ $\hat{i}\times\hat{j} = \hat{k}$; $\hat{j}\times\hat{k} = \hat{i}$; $\hat{k} + \hat{i} = \hat{j}$.
- For a scalar λ, $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a}\times\vec{b}) = (\vec{a}\times\lambda\vec{b})$, where $\vec{a}$ and $\vec{b}$ are given vectors.

SOLVED PROBLEMS

Ex.1 ***Find the projection of $\vec{a} = \hat{i} - 3\hat{k}$ on $\vec{b} = 3\hat{i} + \hat{j} - 4\hat{k}$.***

Sol. Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{3+0+12}{\sqrt{9+1+16}} = \frac{15}{\sqrt{26}}$.

Ex.2 ***If $\vec{a}$ is a unit vector and $(\vec{x} + \vec{a}).(\vec{x} - \vec{a}) = 15$, find $|\vec{x}|$.***

Sol. $\vec{x}^2 - \vec{a}^2 = 15 \Rightarrow |\vec{x}|^2 = 15 + |\vec{a}|^2 = 16 \Rightarrow |\vec{x}| = 4$

Ex.3 ***Find a vector in the direction of vector $5\hat{i} - \hat{j} - 2\hat{k}$ which has magnitude 8 units.***

Sol. The unit vector in the direction of vector $\vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}$ is

$$\frac{1}{|\vec{a}|} \cdot \vec{a} = \frac{1}{\sqrt{5^2 + (-1)^2 + 2^2}} (5\hat{i} - \hat{j} + 2\hat{k})$$

$$= \frac{1}{\sqrt{25+1+4}} (5\hat{i} - \hat{j} + 2\hat{k}) = \frac{1}{\sqrt{30}} (5\hat{i} - \hat{j} + 2\hat{k})$$

The vector of 8 units in the direction of the vector $\vec{a} = \frac{8}{\sqrt{30}} (5\hat{i} - \hat{j} + 2\hat{k})$

Ex.4 ***Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.***

Sol. The resultant of vector $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ is given by

$$\vec{a} + \vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k}) = 3\hat{i} + \hat{j}$$

The unit vector in the direction of $\vec{a} + \vec{b}$ is

$$= \frac{3}{\sqrt{3^2 + 1^2}}\hat{i} + \frac{1}{\sqrt{3^2 + 1^2}}\hat{j} \text{ i.e., } = \frac{3}{\sqrt{10}}\hat{i} + \frac{1}{\sqrt{10}}\hat{j}$$

A vector of magnitude 5 units, and parallel to $\vec{a} + \vec{b}$ is, therefore, $= 5\left(\frac{3}{\sqrt{10}}\hat{i} + \frac{1}{\sqrt{10}}\hat{j}\right)$

i.e., $= \frac{3}{2}\sqrt{10}\,\hat{i} + \frac{\sqrt{10}}{2}\hat{j}$

Ex.5 ***If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.***

Sol. Here, $2\vec{a} = 2\hat{i} + 2\hat{j} + 2\hat{k}$ $\quad -\vec{b} = -2\hat{i} + \hat{j} + 3\hat{k}$

and $3\vec{c} = 3\hat{i} - 6\hat{j} + 3\hat{k}$

$2\vec{a} - \vec{b} + 3\vec{c} = 3\hat{i} - 3\hat{j} + 2\hat{k}$

The unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$

$$= \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}$$

Ex.6 ***Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} - 6\hat{j} - 8\hat{k}$ are collinear.***

Sol. Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} - b_2\hat{j} + b_3\hat{k}$ are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{b_3}{b_3}$

For given vectors, we have $\frac{2}{-4} = \frac{-3}{6} = \frac{4}{-8}$

Each is equal to $\frac{-1}{2}$. Hence, the given vectors are collinear.

Ex.7 ***Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.***

Sol. For a vector $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, a_1, a_2, a_3 are the direction ratios of the vector.
Hence, for the given vector,
$a_1 = 1, a_2 = 2$ and $a_3 = 3$.

$$\frac{1}{\sqrt{1^2 + 2^2 + 3^2}}, \frac{2}{\sqrt{1^2 + 2^2 + 3^2}}, \frac{3}{\sqrt{1^2 + 2^2 + 3^2}}$$

are the required direction cosines.

i.e., $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ are the required direction cosines.

Ex.8 ***Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).***

Sol. The vector joining the initial point (2, 1) and the terminal point (-5, 7) is $= (-5-2)\hat{i} + (7-1)\hat{j}$

$= -7\hat{i} + 6\hat{j}$

The scalar components of the vectors are –7 and 6.

Ex.9 ***Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.***

Sol. The vector $\overrightarrow{PQ} = (4-1)\hat{i} + (5-2)\hat{j} + (6-3)\hat{k}$

$= 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\left|\overrightarrow{PQ}\right| = \left|3\hat{i} + 3\hat{j} + 3\hat{k}\right| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$

so, the unit vector in the direction of PQ

$$= \frac{1}{3\sqrt{3}}.\left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Ex.10 Find the direction cosines of the vector joining the points A(1, 2, –3) and B(–1, –2, 1), directed from A and B.

Sol. The vector $\overrightarrow{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

The direction ratios of $\overrightarrow{AB}$ are, therefore, –2, –4 and 4 and, the direction cosines are

$$\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}$$

Ex.11 Find the position vector of the mid-point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

Sol. The position vectors of the points P and Q are

$2\hat{i} + 3\hat{j} + 4\hat{k}$ and $4\hat{i} + \hat{j} - 2\hat{k}$

The position vector of the mid-point of the vector PQ is $\dfrac{\vec{b} + \vec{a}}{1+1}$

$$= \frac{1}{2}\left(\vec{a} + \vec{b}\right) = \frac{1}{2}\left[(2+4)\hat{i} + (3+1)\hat{j} + (4-2)\hat{k}\right]$$

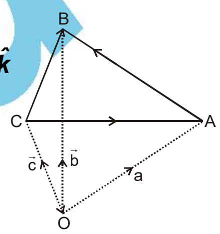
$$= \frac{1}{2}\left[6\hat{i} + 4\hat{j} + 2\hat{k}\right] = 3\hat{i} + 2\hat{j} + \hat{k}$$

Ex.12 Show that the points A, B and C with position vectors

$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$

and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

respectively form the vertices of a right-angled triangle.



Sol. The vector $\overrightarrow{AB} = -\hat{i} + 3\hat{j} + 5\hat{k}$,

$\overrightarrow{BC} = -\hat{i} - 2\hat{j} - 6\hat{k}$, $\overrightarrow{CA} = 2\hat{i} - \hat{j} + \hat{k}$

Now, $AB = \left|\overrightarrow{AB}\right| = \left|-\hat{i} + 3\hat{j} + 5\hat{k}\right|$

$$= \sqrt{-1^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35} \Rightarrow AB^2 = 35$$

$$BC = \left|\overrightarrow{BC}\right| = \left|-\hat{i} - 2\hat{j} - 6\hat{k}\right|$$

$$= \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41} \Rightarrow BC^2 = 41$$

$$CA = \left|\overrightarrow{CA}\right| = \left|2\hat{i} - \hat{j} + \hat{k}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$\Rightarrow CA^2 = 6$

Since $BC^2 = AB^2 + CA^2$; ΔABC is right- angled triangle, right-angled at A.

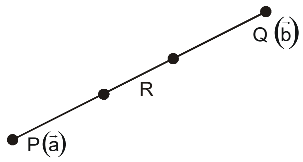
Ex.13 Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

Sol. Let α, β, γ be the angles which the vector $\hat{i} + \hat{j} + \hat{k}$ is inclined to the axes OX, OY and OZ respectively. Then cos α = 1, cos β = 1 and cos γ = 1 $\Rightarrow$ $\alpha = \beta = \gamma$

Ex.14 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1.

Sol. Let the position vector of P be $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and of Q be $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$.

(i) Let R divide the line joining the point P and Q in the ratio 2 : 1 internally. Then, the position vector of R is



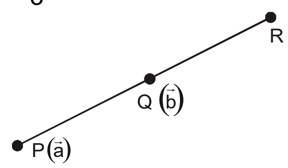
$$\frac{2\vec{b} + \vec{a}}{2+1} = \frac{1}{3}\left(2\vec{b} + \vec{a}\right)$$

$$= \frac{1}{3}\left[\left(-2\hat{i} + 2\hat{j} + 2\hat{k}\right) + \left(\hat{i} + 2\hat{j} - \hat{k}\right)\right]$$

$$= \frac{1}{3}\left[(-2+1)\hat{i} + (2+2)\hat{j} + (2-1)\hat{k}\right] = \frac{1}{3}\left[-\hat{i} + 4\hat{j} + \hat{k}\right]$$

$$= -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(ii) Let R divide the line joining the points P and Q in the ratio 2 :1 externally. Then the position



Vector of R is $\dfrac{2\vec{b} - \vec{a}}{2-1} = 2\vec{b} - \vec{a}$

$$= \left(-2\hat{i} + 2\hat{j} + 2\hat{k}\right) - \left(\hat{i} + 2\hat{j} - \hat{k}\right) = -3\hat{i} + 3\hat{k}$$

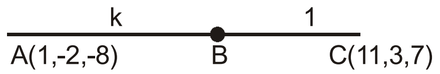
Ex.15 Show that the points A(1, –2, –8), B(5, 0, –2) and C(11, 3, 7) are collinear and find the ratio in which B divides AC.

Sol. Here $\overrightarrow{AB} = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}$

and $\overrightarrow{BC} = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}$

The vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear if $\frac{4}{6} = \frac{2}{3} = \frac{6}{9}$

which is true. Hence, the given points are



collinear. Let B divide AC in the ratio k : 1
Then the coordinates of B are :

$\left(\frac{11k+1}{k+1}, \frac{3k-2}{k+1}, \frac{7k-8}{k+1}\right)$

Equating $\left(\frac{11k+1}{k+1}, \frac{3k-2}{k+1}, \frac{7k-8}{k+1}\right)$ to (5, 0, 2), we

have $\frac{11k+1}{k+1} = 5$; $\frac{3k-2}{k+1} = 0$; $\frac{7k-8}{k+1} = -2$

which is true for $k = \frac{2}{3}$

Hence, B divides AC internally in the ratio 2 : 3.

Ex.16 Show that the following points are collinear: $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$

Sol. $\overrightarrow{AB} = (1+2)\hat{i} + (2-3)\hat{j} + (3-5)\hat{k} = 3\hat{i} - \hat{j} - 2\hat{k}$

$\overrightarrow{BC} = (7-1)\hat{i} - 2\hat{j} + (-1-3)\hat{k} = 6\hat{i} - 2\hat{j} - 4\hat{k}$

$\overrightarrow{AC} = (7+2)\hat{i} - 3\hat{j} + (-1-5)\hat{k} = 9\hat{i} - 3\hat{j} - 6\hat{k}$

Further, $\left|\overrightarrow{AB}\right| = \sqrt{9+1+4} = \sqrt{14}$

$\left|\overrightarrow{BC}\right| = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$

$\left|\overrightarrow{AC}\right| = \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}$

Therefore, $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Hence, the points A, B and C are collinear.

Ex.17 If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

Sol. It is given that $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

Now, $\vec{a} + \vec{b} + \vec{c} = 0$ gives $\vec{a} = -(\vec{b} + \vec{c})$

$\therefore \ \vec{a}.\vec{a} = \vec{a}.\left[-(\vec{b} + \vec{c})\right] = -(\vec{a}.\vec{b} + \vec{a}\vec{c})$

$(\because \ \vec{a}\cdot\vec{a} = |\vec{a}|^2 = 1^2 = 1)$

$\Rightarrow \vec{a}.\vec{b} + \vec{a}.\vec{c} = -1 \Rightarrow \vec{a}.\vec{b} + \vec{c}.\vec{a} = -1 \ (\because \vec{a}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

Similarly, $\vec{b}.\vec{c} + \vec{a}.\vec{b} = -1$ and $\vec{c}.\vec{a} + \vec{b}.\vec{c} = -1$

Adding the three, we get $2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = -3$

$\Rightarrow \ \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -\frac{3}{2}$

Ex.18 If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}.\vec{b} = 0$. But the converse need not be true. Justify your answer with an example.

Sol. If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, the $\vec{a}.\vec{b} = 0$

But the converse is not necessarily true. For, take $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 3\hat{j} - \hat{k}$

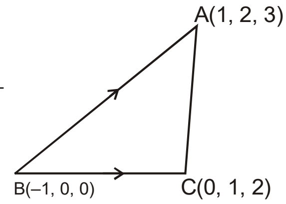
Then $\vec{a}.\vec{b} = 0$. because $\vec{a}.\vec{b} = 6 - 3 - 3 = 0$

Yet. the vectors are non-zero vector.

Ex.19 If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find $\angle ABC$.

Sol. Here, $|\overrightarrow{BA}| = (1+1)\hat{i} + (2-0)\hat{j} + (3-0)\hat{k}$

$= 2\hat{i} + 2\hat{j} + 3\hat{k}$



$\Rightarrow \ |\overrightarrow{BA}| = \sqrt{2^2 + 2^2 + 3^2}$

$\sqrt{4+4+9} = \sqrt{17}$

and $|\overrightarrow{BC}| = (0+1)\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$ So,

$\overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}|.|\overrightarrow{BC}|.\cos \angle ABC$

$(2\hat{i} + 2\hat{j} + 3\hat{k}).(\hat{i} + \hat{j} + 2\hat{k}) = \sqrt{17}.\sqrt{6} \cos \angle ABC$

$\Rightarrow 2 + 2 + 6 = \sqrt{102} \cos \angle ABC \ \Rightarrow \ \frac{10}{\sqrt{102}} = \cos \angle ABC$

$\Rightarrow \ \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$

Ex.20 The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

Sol. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

Then $\vec{b}+\vec{c}=\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$

$=(2+\lambda)\hat{i}+(4+2)\hat{j}+(-5+3)\hat{k} \;=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

$\Rightarrow$ Unit vector along

$$\vec{b}+\vec{c}=\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=\frac{1}{\sqrt{(2+\lambda)^2+36+4}}\left[(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}\right]$$

$$=\frac{2+\lambda}{\sqrt{(2+\lambda)^2+40}}\hat{i}+\frac{6}{\sqrt{(2+\lambda)^2+40}}\hat{j}-\frac{2\hat{k}}{\sqrt{(2+\lambda)^2+40}}$$

Now, we are given that $\vec{a}$. unit vector along $\vec{b}+\vec{c}=1$

So, $\left(\hat{i}+\hat{j}+\hat{k}\right).\left[\frac{2+\lambda}{\sqrt{(2+\lambda)^2+40}}\hat{i}+\frac{6}{\sqrt{(2+\lambda)^2+40}}\hat{j}-\frac{2\hat{k}}{\sqrt{(2+\lambda)^2+40}}\right]=1$

$$\Rightarrow \frac{2+\lambda}{\sqrt{(2+\lambda)^2+40}}+\frac{6}{\sqrt{(2+\lambda)^2+40}}-\frac{2}{\sqrt{(2+\lambda)2+40}}=1$$

$$\Rightarrow \frac{2+\lambda+6-2}{\sqrt{(2+\lambda)^2+40}}=1 \Rightarrow \lambda+6=\sqrt{(2+\lambda)^2+40}$$

$$\Rightarrow (\lambda+6)^2=(2+\lambda)^2+40$$

$$\Rightarrow \lambda^2+12\lambda+36=\lambda^2+4\lambda+4+40 \;\Rightarrow 8\lambda=8 \Rightarrow \lambda=1$$

Ex.21 ***Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d}$ = 15.***

Sol. Let $\vec{d}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Since vector $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$,

$\vec{d}.\vec{a}=0$ and $\vec{d}.\vec{b}=0$

$\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right).\left(\hat{i}+4\hat{j}+2\hat{k}\right)=0$

$\Rightarrow \; a_1+4a_2+2a_3=0$ .....(1)

$\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right).\left(3\hat{i}-2\hat{j}+7\hat{k}\right)=0$

$3a_1-2a_2+4a_3=0$ .....(2)

Also, $\vec{c}\vec{d}=0 \;\Rightarrow\; \left(2\hat{i}-\hat{j}+4\hat{k}\right).\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)=15$

$\Rightarrow \; 2a_1-a_2+4a_3=15$ .....(3)

Solving a_1, a_2 and a_3 from (1), (2) and (3),

we have $a_1=\frac{160}{3}$, $a_2=-\frac{5}{3}$ and $a_3=-\frac{70}{3}$

Thus, the required vector $\vec{d}$ is $\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$.

Ex.22 ***Show that $(\vec{a}-\vec{b}) \times (\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b})$***

Sol. Consider $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=\left(\vec{a}-\vec{b}\right)\times\vec{a}+\left(\vec{a}-\vec{b}\right)\times\vec{b}$

$=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b}$

$=0+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-0=2\left(\vec{a}\times\vec{b}\right)$

Ex.23 ***Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k}) \times (\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$.***

Sol. Given $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$

$\Rightarrow 2\hat{i}\times\hat{i}+6\hat{j}\times\hat{i}+27\hat{k}\times\hat{i}+2\lambda\hat{i}\times\hat{j}+6\lambda\hat{j}\times\hat{j}+27\lambda\hat{k}\times\hat{j}$

$+2\mu\hat{i}\times\hat{k}+6\mu\hat{j}\times\hat{j}+27\mu\hat{k}\times\hat{k}=\vec{0}$

$\Rightarrow 2\times0-6\hat{i}\times\hat{j}+27\hat{k}\times\hat{i}+2\lambda\hat{i}\times\hat{j}+6\lambda\times0-27\lambda\hat{j}\times\hat{k}-2\mu\hat{k}\times\hat{i}$

$+6\mu\hat{j}\times\hat{k}+27\times0=\vec{0}$

$\Rightarrow \; -6\hat{k}+27\hat{j}+2\lambda\hat{k}-27\lambda\hat{i}-2\mu\hat{j}+6\mu\hat{i}=0$

$\Rightarrow \; (6\mu-27\lambda)\hat{i}+(27-2\mu)\hat{j}+(2\lambda-6)\hat{k}=0$

$\Rightarrow \; 6\mu-27\lambda=0;\; 27-2\mu=0$ and $2\lambda-6=0$

$\Rightarrow \; \lambda=3,\; \mu=\frac{27}{2}$

Hence, $\lambda=3$ and $\mu=\frac{27}{2}$.

Ex.24 ***Given that $\vec{a}.\vec{b}$ = 0 and $\vec{a}\times\vec{b}$ = $\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?***

Sol. Given $\vec{a}\cdot\vec{b}$ = 0, we have $|\vec{a}|\,|\vec{b}|\cos\theta = 0$

Either $|\vec{a}| = 0$, $|\vec{b}| = 0$ or $\theta = 90^o$

Given $\vec{a}\times\vec{b}=\vec{0}$, we have $|\vec{a}|\,|\vec{b}|\sin\theta = 0$

Either $|\vec{a}| = 0$, $|\vec{b}| = 0$ or $\theta = 0$

Hence, From (1) and (2) taken together, we have

Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$

Ex.25 ***If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true ? Justify your answer with an example.***

Sol. The converse not true. take any two non-zero collinear vectors, say, $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$

Then, $\vec{a}\neq\vec{0};\vec{b}\neq\vec{0}$ yet $\vec{a}\times\vec{b}=\vec{0}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 2 & -2 & 4\end{vmatrix}$$

$$=(-4+4)\hat{i}+(4-4)\hat{j}+(-2+2)\hat{k}=\vec{0}$$

EXERCISE – I UNSOLVED PROBLEMS

Q.1 If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}, \vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, find the value of λ.

Q.2 Find the angles which vector $\vec{a}$ makes with vectors $\vec{b}$ and $\vec{c}$, where $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}, \vec{b} = 2\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{c} = 2(\hat{i} - \hat{j} + \hat{k})$.

Q.3 Find the value of λ for which the vectors $\vec{a} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$ are (i) parallel, and (ii) perpendicular.

Q.4 Find the work done by the force $\vec{F} = 2\hat{i} + \hat{j} + 2\hat{k}$ in displacing an object from A(1, 2, –3) to B(3, 1, 2).

Q.5 Find the work done by the forces $\hat{i} - \hat{j} + 2\hat{k}$ and $2\hat{i} + 3\hat{j} + \hat{k}$ in displacing an object from the origin to the point A(2, 1, 4).

Q.6 Find the projection of the vector $\hat{i} + 2\hat{j} + 4\hat{k}$ on the vector $2\hat{i} + 4\hat{j} + \hat{k}$.

Q.7 Find the component of the vector $\hat{i} - \hat{j} + 2\hat{k}$ in the direction of $2\hat{i} + \hat{j} - \hat{k}$.

Q.8 Show that the points whose position vectors are $4\hat{i} - 3\hat{j} + \hat{k}, 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - \hat{j}$ from a right triangle.

Q.9 The adjacent sides of a triangle are $2\hat{i} - \hat{j} + 4\hat{k}$ and $\hat{i} - 2\hat{j} - \hat{k}$. Find the area of the triangle.

Q.10 Find the area of a parallelogram whose adjacent sides are $\hat{i} + 2\hat{j} + 2\hat{k}$ and $2\hat{i} + \hat{j} - \hat{k}$.

Q.11 If A(1, –1, 2), B(2, 3, 1) and C(3, 2, –1) are the vertices of ΔABC, find its area using the vector method.

Q.12 If $\hat{i} + 2\hat{j} - \hat{k}$ and $\hat{i} + \hat{j} - 2\hat{k}$ are the two diagonals of a parallelogram, find its area.

Q.13 If A(2, –1, 1), B(1, 2, 0), C(3, 2, 2) and D(4, –1, 3) are the vertices of a parallelogram ABCD, find its area.

Q.14 If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the vertices of ΔABC, show that the area of ΔABC is $\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$

Q.15 Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$. Interpret the result geometrically.

Q.16 Prove that $|\vec{a} \times \vec{b}| = \sqrt{\vec{a}^2 . \vec{b}^2 - (\vec{a}.\vec{b})^2}$.

Q.17 If $|\vec{a}| = 4$, $|\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 8$, find $\vec{a} . \vec{b}$.

Q.18 Find a unit vector perpendicular to both the vectors $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} - \hat{j} + 2\hat{k}$.

Q.19 Find a vector of magnitude 5 units in a direction perpendicular to both the vectors $\hat{i}+\hat{j}-2\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$.

Q.20 Prove that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

Q.21 For any vector $\vec{a}$, prove that $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$.

Q.22 Find a unit vector perpendicular to the plane of ΔABC where the position vectors of A, B and C are $2\hat{i}-\hat{j}+\hat{k}, \hat{i}+\hat{j}+2\hat{k}$ and $2\hat{i}+3\hat{k}$ respectively.

Q.23 Find the angles of ΔABC when the vertices are A(1, 2, −1), B(2, −1, 1) and C(1, 1, −2).

Q.24 For any vector $\vec{r}$, prove that $\vec{r} = (\vec{r}.\hat{i})\hat{i} + (\vec{r}.\hat{j})\hat{j} + (\vec{r}.\hat{k})\hat{k}$.

Q.25 If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ then prove that $\sin\dfrac{\theta}{2} = \dfrac{1}{2}|\hat{a} - \hat{b}|$.

Q.26 If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Q.27 If $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Q.28 For any two vectors $\vec{a}$ and $\vec{b}$, prove that $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$.

Q.29 If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 7$ and $|\vec{c}| = 5$, find the angle between $\vec{a}$ and $\vec{c}$.

Q.30 If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$ then prove that $2\vec{a} + \vec{b}$ is perpendicular to $\vec{b}$.

Answers

EXERCISE – 1 (UNSOLVED PROBLEMS)

2. 60º, 90º **3.** (i) −1 (ii) 13 **4.** 13 units **5.** 20 units **6.** $\dfrac{2\sqrt{21}}{3}$ **7.** $\dfrac{-1}{\sqrt{6}}$

9. $\dfrac{3\sqrt{14}}{2}$ sq units **10.** $5\sqrt{2}$ sq units **11.** $\dfrac{\sqrt{107}}{2}$ sq units **12.** $\sqrt{11}$ sq units **13.** $6\sqrt{2}$ sq units

17. $\pm 4\sqrt{5}$ **18.** $\pm\dfrac{(\hat{i}-11\hat{j}-7\hat{k})}{\sqrt{171}}$ **19.** $\pm\dfrac{5(\hat{i}+5\hat{j}+3\hat{k})}{\sqrt{35}}$ **22.** $\pm\dfrac{3\hat{i}+2\hat{j}-\hat{k}}{\sqrt{14}}$

23. $\angle A = \cos^{-1}\left(\dfrac{1}{2\sqrt{7}}\right)$, $\angle B = \cos^{-1}\left(\dfrac{13}{14}\right)$, $\angle C = \cos^{-1}\left(\dfrac{1}{2\sqrt{7}}\right)$ **27.** $\vec{\beta}_1 = \dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}$; $\vec{\beta}_2 = \dfrac{1}{2}\hat{i} + \dfrac{3}{2}\hat{j} - 3\hat{k}$ **29.** 60º

EXERCISE – II BOARD PROBLEMS

Q.1 Find λ if $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \lambda\hat{i} - 2\hat{j} + 2\hat{k}$ are perpendicular to each other.

Q.2 Find a unit vector perpendicular to both $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$.

Q.3 Define $\vec{a} \times \vec{b}$ and prove that $|\vec{a}\times\vec{b}|\,|\vec{a}\times\vec{b}| = (\vec{a}.\vec{b})$ tan θ, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Q.4 If three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, prove that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

Q.5 If $\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = i - 2k$ find $|2\vec{b}\times\vec{a}|$.

Q.6 Find a vector whose magnitude is 3 units and which is perpendicular to the following two vectors $\vec{a}$ and $\vec{b}$: $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$

Q.7 In ΔOAB, $\overrightarrow{OA} = 3\hat{i} + 2\hat{j} - k$ and $\overrightarrow{OB} = \hat{i} + 3\hat{j} + k$. Find the area of the triangle.

Q.8 Prove that $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2 . |\vec{b}|^2 - (\vec{a}.\vec{b})^2$.

Q.9 For any two vectors $\vec{a}$ and $\vec{b}$, show that $(1+|\vec{a}|^2)\,(1+|\vec{b}|^2) = (1-\vec{a}.\vec{b})^2 + |\vec{a}+\vec{b}+(\vec{a}\times\vec{b})|^2$

Q.10 If $\vec{a}, \vec{b}$ and $\vec{c}$ are position vectors of points A, B and c, then prove that $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is a vector perpendicular to the plane of ΔABC.

Q.11 Find the value of λ so that the two vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and. $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ are
(i) Parallel **(ii)** Perpendicular to each other

Q.12 Find the unit vector perpendicular to the plane ABC where the position vectors of A, B and C are $2\hat{i} - \hat{j} + \hat{k}, \hat{i} + \hat{j} + 2k$ and $2\hat{i} + 3\hat{k}$ respectively.

Q.13 If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal.

Q.14 Show that the points whose position vectors are $\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$. $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - \hat{j}$ form a right angled triangle.

Q.15 Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 3j - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}$. $\vec{d}$ =1.

Q.16 Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i} + \hat{k}$ and the other is perpendicular to $\vec{b}$.

Q.17 If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitude, show that they are equally inclined to the vector $(\vec{a}+\vec{b}+\vec{c})$.

Q.18 If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, show that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular to each other.

Q.19 Find the angle between the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} - 2\hat{k}$

Q.20 If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$ find a vector $\vec{c}$ such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$.

Q.21 Find the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}, \vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$

Q.22 Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ if $|\vec{a}|=1, |\vec{b}|=4$ and $|\vec{c}|=2$.

Q.23 Find a vector of magnitude 5 units. perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

Q.24 If $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+5\hat{j}. 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of the points A, B, C and D, find the angle between $\vec{AB}$ and $\vec{CD}$. Deduce that $\vec{AB}$ and $\vec{CD}$ are collinear.

Q.25 If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3, |\vec{b}|=5$, and $|\vec{c}|=7$, show that angle between $\vec{a}$ and $\vec{b}$ is 60º.

Q.26 The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\lambda\hat{i}+2\hat{j}+3\hat{k}$ $2\hat{i}+4\hat{j}+5\hat{k}$ and is equal to one. Find the value of λ.

Q.27 If $\vec{p}$ is a unit vector and $(\vec{x}-\vec{p})(\vec{x}+\vec{p})=80$, then find $|\vec{x}|$.

Q.28 Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a}.\vec{b}=8$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$

Q.29 If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Q.30 Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=18$.

Q.31 Using vectors, find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Q.32 Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}.\vec{c}=18$.

Q.32 If $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors

Answers

EXERCISE – 2 (BOARD PROBLEMS)

1. –1 **2.** $\frac{1}{5\sqrt{3}}(5\hat{i}-\hat{j}+7\hat{k})$ **6.** $2\hat{i}-2\hat{j}+\hat{k}$ **7.** $\frac{3}{2}\sqrt{10}$ sq units **10.** 60 **27.** $6\sqrt{14}$ **11. (i)** –2 **(ii)** 26

12. $\frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}-\hat{k}$ **15.** $\frac{1}{4}(\hat{i}+\hat{j}+3\hat{k})$ **16.** $6\hat{i}+2\hat{k}; -\hat{i}-2\hat{j}+3\hat{k}$ **19.** $\frac{\pi}{2}$ **20.** $\frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$ **21.** 2

22. $-\frac{21}{2}$ **23.** $-\frac{5}{\sqrt{6}}\hat{i}+\frac{10}{\sqrt{6}}\hat{j}-\frac{5}{\sqrt{6}}\hat{k}$ **24.** 180º **26.** 1 **27.** 9 **28.** $\frac{8}{7}$ **29.** $2\hat{i}-4\hat{j}+4\hat{k}$ **30.** $64\hat{i}-2\hat{j}-28\hat{k}$

31. $\frac{1}{2}\sqrt{61}$ **32.** $\vec{p}=64\hat{i}-2\hat{j}-28\hat{k}$