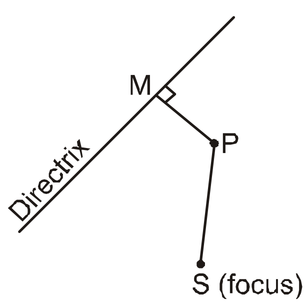
Hyperbola

Hyperbolic curves are of special importance in the field of science and technology especially astronomy and space studies. In this chapter we are going to study the characteristics of such curves.

DEFINITION

A hyperbola is defined as the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point to that from a fixed line (the point does not lie on the line) is

a fixed constant greater than 1.



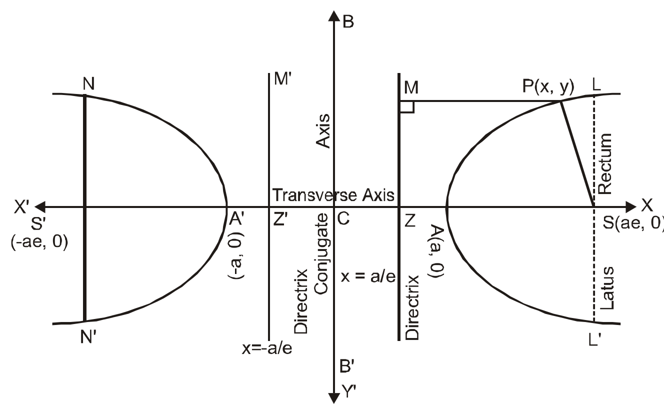
$\frac{PS}{PM} = e > 1$, e – eccentricity

EQUATION OF HYPERBOLA IN STANDARD FORM

The general form of standard hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are constants.



Standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

where $b^2 = a^2 (e^2 - 1)$.

* **Eccentricity (e) :**

$$e^2 = 1 + \frac{b^2}{a^2}$$

* **Foci :**

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

* **Equations of directrices :**

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

* **Transverse axis :**

The line segment A′A of length 2a in which the foci S′ & S both lie is called the transverse axis of the hyperbola.

* **Conjugate axis :**

The line segment B′B of length 2b between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola.

* **Principal axes :**

The transverse & conjugate axis together are called principal axes of the hyperbola.

* **Vertices :**

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$

* **Focal chord :**

A chord which passes through a focus is called a focal chord.

* **Double ordinate :**

A chord perpendicular to the transverse axis is called a double ordinate.

* **Latus rectum :**

Focal chord perpendicular to the transverse axis is called latus rectum.

Its length (ℓ) is given by

$$\ell = \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a(e^2 - 1).$$

Note :

(i) Length of latus rectum = 2 e × (distance of focus from corresponding directrix)

(ii) End points of latus rectum are $L \equiv \left(ae, \frac{b^2}{a}\right)$,

$L' \equiv \left(ae, -\frac{b^2}{a}\right), M \equiv \left(-ae, \frac{b^2}{a}\right), M' \equiv \left(-ae, -\frac{b^2}{a}\right)$

* **Centre :**

The point which bisects every chord of the conic, drawn through it, is called the centre of the conic. $C \equiv (0,0)$ the origin is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

General note :

Since the fundamental equation to hyperbola only differs from that to ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for hyperbola are derived from those for ellipse by simply changing the sign of b^2.

Solved Examples

Ex.1 Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus (1, 2) and eccentricity $\sqrt{3}$.

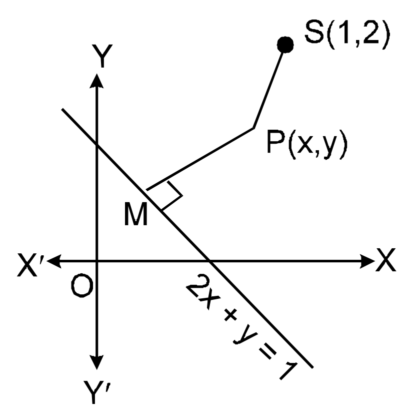
Sol. Let P(x,y) be any point on the hyperbola.

Draw PM perpendicular from P on the directrix.

Then by definition SP = e PM

$\Rightarrow \quad (SP)^2 = e^2 (PM)^2$

$\Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x + y - 1}{\sqrt{4+1}}\right\}^2$



$\Rightarrow 5\{x^2 + y^2 - 2x - 4y + 5\}$

$= 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Ex.2 Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Sol. Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Then transverse axis = 2a and latus–rectum = $\frac{2b^2}{a}$

According to question $\frac{2b^2}{a} = \frac{1}{2}(2a)$

$\Rightarrow 2b^2 = a^2 \qquad (\because b^2 = a^2(e^2 - 1))$

$\Rightarrow 2a^2(e^2 - 1) = a^2 \qquad \Rightarrow \qquad 2e^2 - 2 = 1$

$\Rightarrow \quad e^2 = \frac{3}{2} \qquad \therefore \qquad e = \sqrt{\frac{3}{2}}$

Hence the required eccentricity is $\sqrt{\frac{3}{2}}$.

CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

is $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \left(\text{i.e., } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1\right)$

Note :

(a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similar if they have the same eccentricity.

(d) Two similar hyperbolas are said to be equal if they have same latus rectum.

(e) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

PROPERTIES OF HYPERBOLA AND ITS CONJUGATE

	Hyperbola	**Conjugate Hyperbola**
Standard equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{-x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	(0, 0)	(0, 0)
Equation of transverse axis	y = 0	x = 0
Equation of conjugate axis	x = 0	y = 0
Length of transverse axis	2a	2b
Length of Conjugate axis	2b	2a
Foci	(± ae, 0)	(0, ± be)
Equation of directrices	x = ± a/e	y = ± b/e
Vertices	(± a, 0)	(0, ± b)
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$	$e = \sqrt{\frac{a^2 + b^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parameter Coordinates	$(a \sec\theta, b \tan\theta)$	$(b \sec\theta, a \tan\theta)$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii (S'P – SP)	2a	2b
Tangent at the vertices	x = ± a	y = ± b

Solved Examples

Ex.3 Find the eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$.

Sol. We have $x^2 - y^2 - 4x + 4y + 16 = 0$

$\Rightarrow (x^2 - 4x) - (y^2 - 4y) = 16$

$\Rightarrow (x^2 - 4x + 4) - (y^2 - 4y + 4) = -16$

$\Rightarrow (x-2)^2 - (y-2)^2 = -16$

$$\Rightarrow \frac{(x-2)^2}{4^2} - \frac{(y-2)^2}{4^2} = 1$$

Shifting the origin at (2, 2), we obtain $\frac{X^2}{4^2} - \frac{Y^2}{4^2} = -1$, where $x = X + 2, y + Y + 2$

This is rectangular hyperbola, whose eccentricity is always $\sqrt{2}$.

Ex.4 Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus (1, 2) and eccentricity $\sqrt{3}$.

Sol. Let P(x, y) be any point on the hyperbola and PM is perpendicular from P on the directrix.

Then by definition

SP = e PM

$\Rightarrow (SP)^2 = e^2 (PM)^2$

$$\Rightarrow (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x+y-1}{\sqrt{4+1}}\right\}^2$$

$\Rightarrow 5(x^2 + y^2 - 2x - 4y + 5) = 3$
$(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12xy - 2x + 9y - 22 = 0$

which is the required hyperbola.

Ex.5 Find the eccentricity of the hyperbola $16x^2 - 32x - 3y^2 + 12y = 44$.

Sol. We have, $16(x^2 - 2x) - 3(y^2 - 4y) = 44$

$\Rightarrow 16(x-1)^2 - 3(y-2)^2 = 48$

$$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1$$

This equation represents a hyperbola with eccentricty given

$$e = \sqrt{1 + \left(\frac{\text{Conjugate axis}}{\text{Transverse axis}}\right)} = \sqrt{1 + \left(\frac{4}{\sqrt{3}}\right)^2} = \sqrt{\frac{19}{3}}$$

Ex.6 Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, length of the latus-rectum and equations of the directrices of the following hyperbola

$16x^2 - 9y^2 = -144$.

Sol. The equation $16x^2 - 9y^2 = -144$ can be written as

$$\frac{x^2}{9} - \frac{y^2}{16} = -1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore a^2 = 9, b^2 = 16$

$\Rightarrow a = 3, b = 4$

Length of transverse axis :

The length of transverse axis = 2b = 8

Length of conjugate axis :

The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$

Foci : The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

Vertices : The co–ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$

Length of latus–rectum :

The length of latus–rectum $= \frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices :

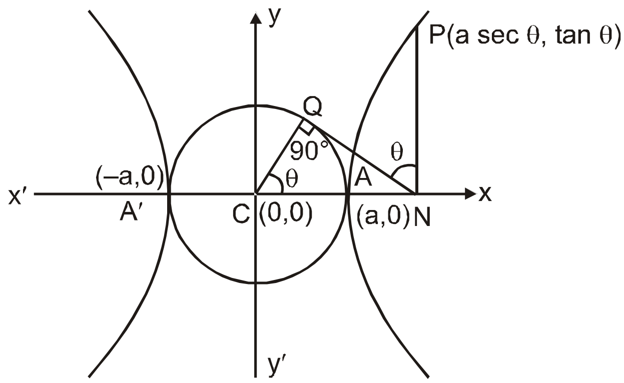
The equation of directrices are

$$y = \pm \frac{b}{e} \Rightarrow y = \pm \frac{4}{(5/4)} \Rightarrow y = \pm \frac{16}{5}$$

Auxiliary Circle :

A circle drawn with centre C and transverse axis as a diameter is called the **auxiliary circle** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the following figure that P & Q are called the **"corresponding points"** of the hyperbola & the auxiliary circle.



PARAMETRIC EQUATIONS OF THE HYPERBOLA

Since coordinates x = a sec θ and y = b tan θ satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for all real values of θ therefore, x = a sec θ, y = b tan θ are the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the parameter $0 \le \theta < 2\pi$.

Hence, the coordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as (a sec θ, b tan θ). This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point (a secθ, b tanθ) on the hyperbola.

Equation of Chord The equation of the chord joining the points

$P \equiv (a \sec\theta_1, b \tan\theta_1)$ and $Q \equiv (a \sec \theta_2, b \tan \theta_2)$ is

$$\frac{x}{a}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1 + \theta_2}{2}\right) = \cos\left(\frac{\theta_1 + \theta_2}{2}\right)$$

or $$\begin{vmatrix} x & y & 1 \\ a\sec\theta_1 & b\tan\theta_1 & 1 \\ a\sec\theta_2 & b\tan\theta_2 & 1 \end{vmatrix} = 0$$

POSITION OF A POINT WITH RESPECT TO A HYPERBOLA

The point $P(x_1, y_1)$ lies outside, on or inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0$. $= 0$ or < 0.

Solved Examples

Ex.7 Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Sol. Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$,

So the point (5,–4) lies inside the hyperbola $9x^2 - y^2 = 1$.

CONDITION FOR TANGENCY AND POINTS OF CONTACT

The condition for the line y = mx + c to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$$

Solved Examples

Ex.8 Show that the line x cos α + y sin α = p touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Sol. The given line is

$x \cos\alpha + y \sin\alpha = p \Rightarrow y \sin\alpha = - x \cos\alpha + p$

$\Rightarrow y = - x \cot\alpha + p \operatorname{cosec}\alpha$

Comparing this line with y = mx + c

$m = - \cot\alpha$, $c = p \operatorname{cosec}\alpha$

Since the given line touches trhe hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2 \Rightarrow p^2 \operatorname{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Tangents :

(i) Slope form : $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having slope 'm'.

(ii) Point form : Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(iii) Parametric form : Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point.

$(a \sec\theta, b \tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.

Note :

(i) Point of intersection of the tangents at $P(\theta_1)$ & $Q(\theta_2)$

is $\left(a\frac{\cos\frac{\theta_1 - \theta_2}{2}}{\cos\frac{\theta_1 + \theta_2}{2}},\ b\tan\left(\frac{\theta_1 + \theta_2}{2}\right) \right)$

(ii) If $|\theta_1 + \theta_2| = \pi$, then tangents at these points $(\theta_1$ & $\theta_2)$ are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touches the hyperbola at the extremities of a diameter.

Notes :

* **Number of Tangents From a Point** Two tangents can be drawn from a point to a hyperbola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the hyperbola.

* **Director Circle** It is the locus of points from which $\perp$ tangents are drawn to the hyperbola. The equation of director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$.

Solved Examples

Ex.9 Prove that the straight line $\ell x + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\ell^2 - b^2m^2 = n^2$.

Sol. The given line is $\ell x + my + n = 0$ or

$y = -\frac{\ell}{m}x - \frac{n}{m}$

Comparing this line with $y = Mx + c$

$\therefore\ M = -\frac{\ell}{m}$ and $c = -\frac{n}{m}$ ....(1)

This line (1) will touch the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2M^2 - b^2$

$\Rightarrow \frac{n^2}{m^2} = \frac{a^2\ell^2}{m^2} - b^2$ or $a^2\ell^2 - b^2m^2 = n^2$

Ex.10 Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore\ m \times 1 = -1 \quad \Rightarrow \quad m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore\ a^2 = 36$ and $b^2 = 9$

So the equation of tangents are

$y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27} \quad \Rightarrow \quad x + y \pm 3\sqrt{3} = 0$

Ex.11 Find the equation and the length of the common tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Sol. Tangent at ($a\sec\phi$, $b\tan\phi$) on the 1st hyperbola is

$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$ .....(1)

Similarly tangent at any point ($b\tan\theta$, $a\sec\theta$) on 2nd hyperbola is

$\frac{y}{a}\sec\theta - \frac{x}{b}\tan\theta = 1$ .....(2)

If (1) and (2) are common tangents then they should be identical. Comparing the co–effecients of x and y

$\Rightarrow \frac{\sec\theta}{a} = -\frac{\tan\phi}{b}$ .....(3)

and $-\frac{\tan\theta}{b} = \frac{\sec\phi}{a}$

or $\sec\theta = -\frac{a}{b}\tan\phi$ ......(4)

$\because \sec^2\theta - \tan^2\theta = 1$

$\Rightarrow \frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\sec^2\phi = 1$ {from (3) and (4)}

or $\frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}(1+\tan^2\phi) = 1$

or $\left(\frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\tan^2\phi = 1 + \frac{b^2}{a^2}$

$\tan^2\phi = \frac{b^2}{a^2 - b^2}$

and $\sec^2\phi = 1 + \tan^2\phi = \frac{a^2}{a^2 - b^2}$

Hence the point of contact are

$\left\{\pm\frac{a^2}{\sqrt{(a^2-b^2)}}, \pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right\}$ and

$\left\{\pm\frac{b^2}{\sqrt{(a^2-b^2)}}, \pm\frac{a^2}{\sqrt{(a^2-b^2)}}\right\}$ {from (3) and (4)}

Length of common tangent i.e., the distance between the above points is $\sqrt{2}\frac{(a^2+b^2)}{\sqrt{(a^2-b^2)}}$ and equation of common tangent on putting the values of $\sec\phi$ and $\tan\phi$ in (1) is

$\pm\frac{x}{\sqrt{(a^2-b^2)}} \mp \frac{y}{\sqrt{(a^2-b^2)}} = 1$ or

$x \mp y = \pm\sqrt{(a^2-b^2)}$

EQUATION OF NORMAL IN DIFFERENT FORMS

Point Form The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$

Parametric Form The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point ($a\sec\theta$, $b\tan\theta$) is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

Slope Form The equation of normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of slope 'm' is

$y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}}$

Notes :

* The coordinates of the points of contact are

$\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$

* **Number of Normals** In general, four normals can be drawn to a hyperbola from a point in its plane i.e., there are four points on the hyperbola, the normals at which will pass through a given point. These four points are called the co-normal points.
* Tangent drawn at any point bisects the angle between the lines joining the point to the foci, whereas normal bisects the supplementary angle between the lines.

Solved Examples

Ex.12 A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

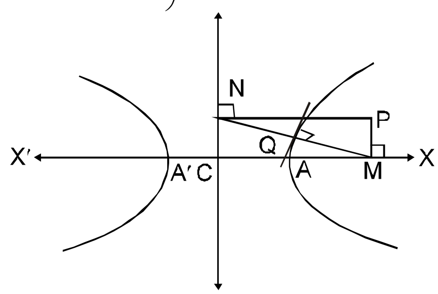
Sol. The equation of normal at the point Q $(a \sec\phi, b \tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$ax \cos\phi + by \cot\phi = a^2 + b^2$ ....(1)

The normal (1) meets the x–axis in

$M\left(\frac{a^2+b^2}{a}\sec\phi,\ 0\right)$ and y–axis in

$N\left(0, \frac{a^2+b^2}{b}\tan\phi\right)$



$\therefore$ Equation of MP, the line through M and perpendicular to x–axis, is

$x = \left(\frac{a^2+b^2}{a}\right) \sec\phi$ or $\sec\phi = \frac{ax}{(a^2+b^2)}$ ....(2)

and the equation of NP, the line through N and perpendicular to the y–axis is

$y = \left(\frac{a^2+b^2}{b}\right) \tan\phi$ or $\tan\phi = \frac{by}{(a^2+b^2)}$ ....(3)

The locus of the point of intersection of MP and NP will be obtained by eliminating ϕ from (2) and (3), we have $\sec^2\phi - \tan^2\phi = 1$

$\Rightarrow \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = 1$ or

$a^2x^2 - b^2y^2 = (a^2 + b^2)^2$ is the required locus of P.

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point

$P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $SS_1 = T^2$

where $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

and $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Solved Examples

Ex.13 How many real tangents can be drawn from the point (4, 3) to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

Sol. Given point $P \equiv (4, 3)$

Hyperbola $S \equiv \frac{x^2}{16} - \frac{y^2}{9} - 1 = 0$

$\because S_1 \equiv \frac{16}{16} - \frac{9}{9} - 1 = -1 < 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the hyperbola.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is $SS_1 = T^2$

$\Rightarrow \left(\frac{x^2}{16} - \frac{y^2}{9} - 1\right) . (-1) = \left(\frac{4x}{16} - \frac{3y}{9} - 1\right)^2$

$\Rightarrow -\frac{x^2}{16} + \frac{y^2}{9} + 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 - \frac{xy}{6} - \frac{x}{2} + \frac{2y}{3}$

$\Rightarrow 3x^2 - 4xy - 12x + 16y = 0$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

Ex.14 Find the locus of point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Sol. Let P(h, k) be the point of intersection of two perpendicular tangents. Equation of pair of tangents is $SS_1 = T^2$

$\Rightarrow \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} - \frac{k^2}{b^2} - 1\right) = \left(\frac{hx}{a^2} - \frac{ky}{b^2} - 1\right)^2$

$\Rightarrow \frac{x^2}{a^2}\left(-\frac{k^2}{b^2} - 1\right) - \frac{y^2}{b^2}\left(\frac{h^2}{a^2} - 1\right) + ... = 0$......(i)

Since equation (i) represents two perpendicular lines

$\therefore \frac{1}{a^2}\left(-\frac{k^2}{b^2} - 1\right) - \frac{1}{b^2}\left(\frac{h^2}{a^2} - 1\right) = 0$

$\Rightarrow -k^2 - b^2 - h^2 + a^2 = 0$

$\Rightarrow$ locus is $x^2 + y^2 = a^2 - b^2$

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is T = 0, where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

Solved Examples

Ex.15 If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Sol. Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is

$$\frac{xh}{a^2}-\frac{yk}{b^2}=1 \quad ....(i)$$

which touches the parabola

equation of tangent to parabola $y^2 = 4ax$

$$y = mx + \frac{a}{m} \quad \Rightarrow mx - y = -\frac{a}{m} \quad ....(ii)$$

equation (i) & (ii) as must be same

$$\therefore \frac{m}{\left(\frac{h}{a^2}\right)}=\frac{-1}{\left(-\frac{k}{b^2}\right)}=\frac{-\frac{a}{m}}{1}$$

$$\Rightarrow m = \frac{h}{k}\frac{b^2}{a^2} \ \& \ m = -\frac{ak}{b^2}$$

$$\therefore \frac{hb^2}{ka^2}=-\frac{ak}{b^2} \quad \Rightarrow \text{ locus of P is } y^2 = -\frac{b^4}{a^3}.x$$

CHORD WITH A GIVEN MID POINT

The equation of chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with $P(x_1, y_1)$ as its middle point is given by $T = S_1$ where

$$T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 \text{ and } S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

Solved Examples

Ex.16 Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$, which are tangent to the hyperbola $9x^2 - 16y^2 = 144$.

Sol. Any tanget to hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ is

$$y = mx + \sqrt{(16m^2-9)} \quad ....(i)$$

Let (x_1, y_1) be the mid-point of the chord of the circel $x^2 + y^2 = 16$, then equation of the chord is $(T = S_1)$

$$xx_1 + yy_1 - (x_1^{\ 2} + y_1^{\ 2}) = 0 \quad ....(ii)$$

Since (i) and (ii) are same, comparing, we get

$$\frac{m}{x_1}=-\frac{1}{y_1}=\frac{\sqrt{16m^2-9}}{-(x_1^2+y_1^2)}$$

$$\Rightarrow m = -\frac{x_1}{y_1}, (x_1^{\ 2} + y_1^{\ 2})^2 = y_1^{\ 2}(15m^2 - 9)$$

Eliminating m and generalizing (x_1, y_1) required locus is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.

Ex.17 Find the locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

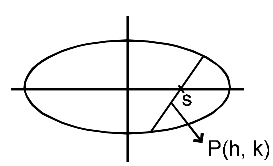
Sol. Let $P \equiv (h, k)$ be the mid-point

$\therefore$ equation of chord whose mid-point is given is

$$\frac{xh}{a^2}-\frac{yk}{b^2}-1=\frac{h^2}{a^2}-\frac{k^2}{b^2}-1$$

since it is a focal chord,

$\therefore$ it passes through focus, either (ae, 0) or (–ae, 0)



If it passes through (ae, 0)

$$\therefore \text{ locus is } \frac{ex}{a}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

If it passes through (–ae, 0)

$$\therefore \text{ locus is } -\frac{ex}{a}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

Ex.18 Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1$ passing through (a, b) are bisected by the line $x + y = b$.

Sol. Let the line $x + y = b$ bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ equation of chord whose mid-point is $P(\alpha, b - \alpha)$

$$\frac{x\alpha}{2a^2} - \frac{y(b-\alpha)}{2b^2} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b)

$$\therefore \frac{\alpha}{2a} - \frac{(b-\alpha)}{2b} = \frac{\alpha^2}{2a^2} - \frac{(b-\alpha)^2}{2b^2}$$

$$\alpha^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + \alpha\left(\frac{1}{b} - \frac{1}{a}\right) = 0$$

$$\alpha = 0, \quad \alpha = \frac{1}{\frac{1}{a} + \frac{1}{b}} \qquad \therefore \qquad a \neq \pm b$$

Ex.19 Find the locus of the mid point of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend a right angle at the origin.

Sol. let (h,k) be the mid–point of the chord of the hyperbola. Then its equation is

$$\frac{hx}{a^2} - \frac{ky}{b^2} - 1 = \frac{h^2}{b^2} - \frac{k^2}{b^2} - 1 \quad \text{or}$$

$$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \qquad \text{........(1)}$$

The equation of the lines joining the origin to the points of intersection of the hyperbola and the chord (1) is obtained by making homogeneous hyperbola with the help of (1)

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2$$

$$y^2 = \frac{h^2}{a^4}x^2 + \frac{k^2}{b^4}y^2 - \frac{2hk}{a^2b^2}xy \qquad \text{.......(2)}$$

The lines represented by (2) will be at right angle if coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2$$

$$- \frac{k^2}{b^4} = 0$$

$$\Rightarrow \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{h^2}{a^4} + \frac{k^2}{b^4}$$

hence, the locus of (h,k) is $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$

$$= \frac{x^2}{a^4} + \frac{y^2}{b^4}$$

EQUATION OF A DIAMETER OF A HYPERBOLA

The equation of the diameter bisecting chords of slope m of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2m}$.

CONJUGATE DIAMETERS

Two diameters of a hyperbola are said to be conjugate diameters if each bisects the chord parallel to the other. If m_1 and m_2 be the slopes of the conjugate diameters of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $m_1m_2 = \frac{b^2}{a^2}$

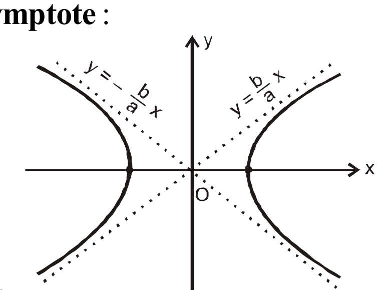
Asymptotes :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called **asymptote** of the hyperbola.

* **Equations of asymptote** :

$$\frac{x}{a} + \frac{y}{b} = 0$$

and $\frac{x}{a} - \frac{y}{b} = 0$.



* **Pair of asymptotes** :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

Note :

(i) A hyperbola and its conjugate have the same asymptote.

(ii) The equation of the pair of asymptotes differs from the equation of hyperbola (or conjugate hyperbola) by the constant term only.

(iii) The asymptotes pass through the centre of the hyperbola & are equally inclined to the transverse axis of the hyperbola. Hence the bisectors of the angles between the asymptotes are the principle axes of the hyperbola.

(iv) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(v) A simple method to find the co-ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:

Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the centre of the hyperbola.

Remarks :

(i) No tangent to the hyperbola can be drawn from its centre.

(ii) Only one tangent to the hyperbola can be drawn from a point lying on its asymptotes other than centre

(iii) Two tangents can be drawn to the hyperbola from any of its external points which does not lie at its asymptotes

Solved Examples

Ex.20 Find the asymptotes of $xy - 3y - 2x = 0$.

Sol. Since equation of a hyperbola and its asymptotes differ in constant terms only,

$\therefore$ Pair of asymptotes is given by $xy - 3y - 2x + \lambda = 0$

where λ is any constant such that it represents two straight lines.

$\therefore\ abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow 0 + 2 \times -\frac{3}{2} \times -1 \times \frac{1}{2} - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0$$

$\therefore\ \lambda = 6$

From (1), the asymptotes of given hyperbola are given by

$xy - 3y - 2x + 6 = 0$ or $(y - 2)(x - 3) = 0$

$\therefore$ Asymptotes are $x - 3 = 0$ and $y - 2 = 0$

Ex.21 The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3), show that its equation is $(2x + 3y - 8)(3x + 2y + 7) = 154$

Sol. Let the asymptotes be $2x + 3y + \lambda = 0$ and $3x + 2y + \mu = 0$. Since asymptotes passes through (1, 2), then $\lambda = -8$ and $\mu = -7$

Thus the equation of asymptotes are $2x + 3y - 8 = 0$ and $3x + 2y - 7 = 0$

Let the equation of hyperbola be

$(2x + 3y - 8)(3x + 2y - 7) + v = 0$ ......(1)

It passes through (5,3), then

$(10 + 9 - 8)(15 + 6 - 7) + v = 0$

$\Rightarrow\ 11 \times 14 + v = 0$

$\therefore\ v = -154$

putting the value of v in (1) we obtain

$(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$

which is the equation of required hyperbola.

Rectangular hyperbola (equilateral hyperbola) :

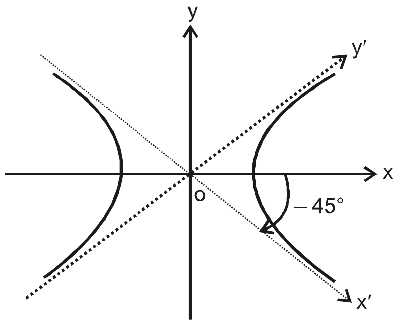
The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$.

Since $a = b$

equation becomes $x^2 - y^2 = a^2$

whose asymptotes are $y = \pm x$.

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$



Rotation of this system through an angle of 45° in clockwise direction gives another form to the equation of rectangular hyperbola.

which is $xy = c^2$ where $c^2 = \frac{a^2}{2}$.

Properties of Rectangular hyperbola $xy = c^2$

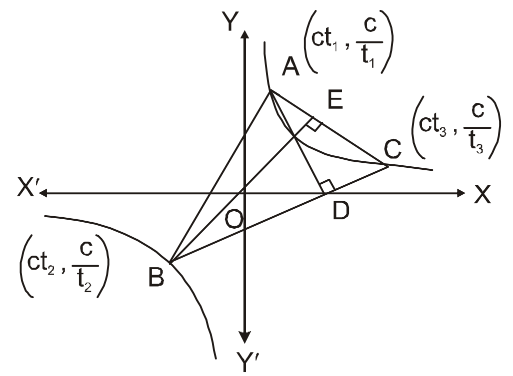
* Equation of the chord joining 't_1' and 't_2' is $x + yt_1t_2 - c(t_1 + t_2) = 0$
* Equation of tangnet at (x_1, y_1) is $xy_1 + x_1y = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$
* Equation of tangent at 't' is : $\frac{x}{t} + yt = 2c$.
* Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1t_2}\right)$
* Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.
* Equation of normal at 't' is: $xt^3 - yt - ct^4 + c = 0$
* The equation of the chord of the hyperbola $xy = c^2$ whose middle point is (x_1, y_1) is $T = S_1$ i.e., $xy_1 + x_1y = 2x_1y_1$.
* The slope of the tangent at the point $(ct, c/t)$ is $-1/t^2$, which is always negative. Hence tangents drawn at any point to $xy = c^2$ would always make an obtuse angle with the x-axis.
* The slope of the normal at the point $(ct, c/t)$ is t^2 which is always positive. Hence normal drawn to $xy = c^2$ at any point would always make an acute angle with the x-axis.

Solved Examples

Ex.22 A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Sol. Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ Co–ordinates of A,B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively



Now slope of BC is $\frac{c(t_3 - t_2)}{c(t_2 - t_3)t_2t_3} = -\frac{1}{t_2t_3}$

$\therefore$ Slope of AD is t_2t_3

Equation of Altitude AD is $y - \frac{c}{t_1} = t_2t_3(x - ct_1)$

or $t_1y - c = x\, t_1t_2t_3 - ct_1^2t_2t_3$ .....(1)

Similarly equation of altitude BE is

$t_2y - c = x\, t_1t_2t_3 - ct_1t_2^2t_3$ ......(2)

Solving (1) and (2), we get the orthocentre $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

Which lies on $xy = c^2$.

Ex.23 A, B, C are three points on the rectangular hyperbola $xy = c^2$, find

(i) The area of the triangle ABC

(ii) The area of the triangle formed by the tangents at A, B and C.

Sol. Let co–ordinates of A,B and C on the hyperbola xy $= c^2$ are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively.

(i) $\therefore$ Area of triangle ABC

$$= \frac{1}{2}\left[\begin{vmatrix} ct_1 & \frac{c}{t_1} \\ ct_2 & \frac{c}{t_2} \end{vmatrix} + \begin{vmatrix} ct_2 & \frac{c}{t_2} \\ ct_3 & \frac{c}{t_3} \end{vmatrix} + \begin{vmatrix} ct_3 & \frac{c}{t_3} \\ ct_1 & \frac{c}{t_1} \end{vmatrix}\right]$$

$$= \frac{c^2}{2}\left|\frac{t_1}{t_2} - \frac{t_2}{t_1} + \frac{t_2}{t_3} - \frac{t_3}{t_2} + \frac{t_3}{t_1} - \frac{t_1}{t_3}\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\left|t_1^2t_3 - t_2^2t_3 + t_1t_2^2 - t_3^2t_1 + t_2t_3^2 - t_1^2t_2\right|$$

$$= \frac{c^2}{2t_1t_2t_3}\ |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|$$

(ii) Equations of tangents at A,B,C are

$x + yt_1^2 - 2ct_1 = 0$

$x + yt_2^2 - 2ct_2 = 0$

and $x + yt_3^2 - 2ct_3 = 0$

$\therefore$ Required Area

$$= \frac{1}{2|C_1C_2C_3|}\begin{vmatrix} 1 & t_1^2 & -2ct_1 \\ 1 & t_2^2 & -2ct_2 \\ 1 & t_3^2 & -2ct_3 \end{vmatrix}^2 \quad ....(1)$$

where $C_1 = \begin{vmatrix} 1 & t_2^2 \\ 1 & t_3^2 \end{vmatrix}$, $C_2 = -\begin{vmatrix} 1 & t_1^2 \\ 1 & t_3^2 \end{vmatrix}$ and

$C_3 = \begin{vmatrix} 1 & t_1^2 \\ 1 & t_2^2 \end{vmatrix}$

$\therefore C_1 = t_3^2 - t_2^2,\ C_2 = t_1^2 - t_3^2$ and $C_3 = t_2^2 - t_1^2$

From (1) $= \dfrac{1}{2|(t_3^2 - t_2^2)(t_1^2 - t_3^2)(t_2^2 - t_1^2)|}$

$4c^2.(t_1 - t_2)^2 (t_2 - t_3)^2 (t_3 - t_1)^2$

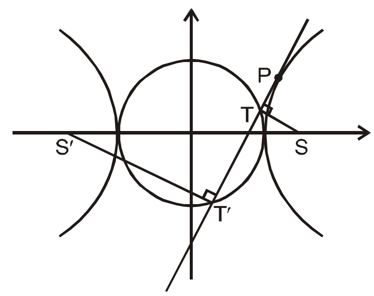
$$= 2c^2 \left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$$

$\therefore$ Required area is, $2c^2$

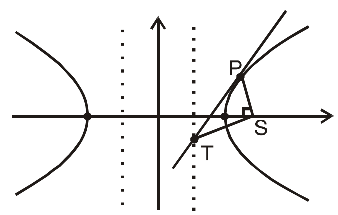
$$\left|\frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_2 + t_3)(t_3 + t_1)}\right|$$

Important results :

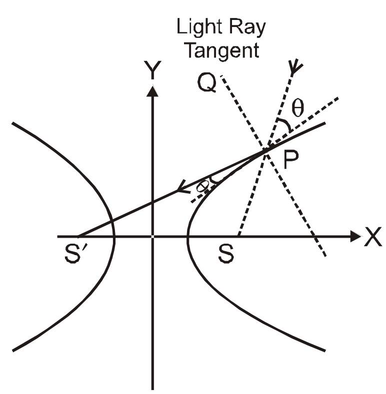
* Difference of focal distances is a constant, i.e. $|PS - PS'| = 2a$
* Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.



* The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

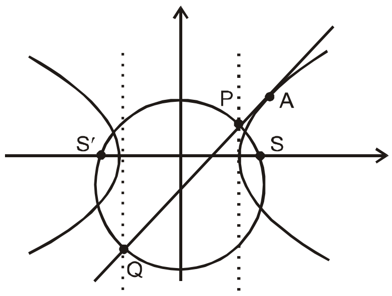


* The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

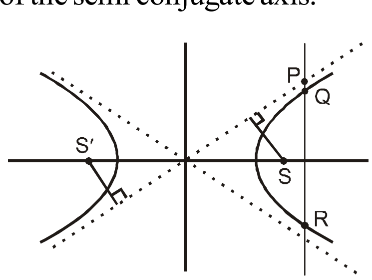


Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1\ (a>k>b>0)$ are confocal and therefore orthogonal.

* The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



* If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.



(PQ) (PR) = b^2

* Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.
* The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2-b^2y^2)=(a^2+b^2)^2$.
* If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.
* A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right)$ i = 1, 2, 3 be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.
* If a circle and the rectangular hyperbola $xy=c^2$ meet in the four points t_1, t_2, t_3 & t_4, then

(a) $t_1t_2t_3t_4=1$

(b) the centre of the mean position of the four points bisects the distance between the centres of the two curves.

(c) the centre of the circle through the points t_1, t_2 & t_3 is :

$$\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1t_2t_3\right)\right\}$$

Solved Examples

Ex.24 A ray originating from the point (5, 0) is incident on the hyperbola $9x^2-16y^2=144$ at the point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lying in first quadrant.

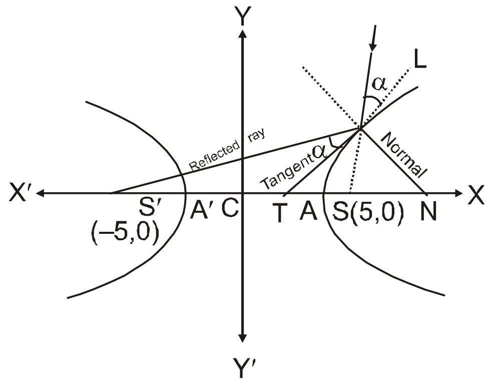
Sol. Given hyperbola is $9x^2-16y^2=144$. This equation can be

rewritten as $\frac{x^2}{16}-\frac{y^2}{9}=1$ ......(1)

Since x co–ordinate of P is 8. Let y co–ordinate of P is α.

$\because$ $(8,\alpha)$ lies on (1)

$\therefore$ $\frac{64}{16}-\frac{\alpha^2}{9}=1$



$\Rightarrow \alpha^2 = 27$

$\Rightarrow \alpha = 3\sqrt{3}$ ($\because$ P lies in first quadrant)

Hence co-ordinate of point P is $(8, 3\sqrt{3})$.

$\because$ Equation of reflected ray passes through P $(8, 3\sqrt{3})$ and S′(–5,0)

$\therefore$ Its equation is $y - 3\sqrt{3} = \dfrac{0 - 3\sqrt{3}}{-5-8}(x - 8)$

or $13y - 39\sqrt{3} = 3\sqrt{3}\,x - 24\sqrt{3}$

or $3\sqrt{3}\,x - 13y + 15\sqrt{3} = 0$.