BINOMIAL THEOREM

GENERAL TERM AND MIDDLE TERM

Pascal's Triangle

A triangular arrangement of numbers as shown. The numbers give the coefficients for the expansion of $(x + y)^n$. The first row is for $n = 0$, the second for $n = 1$, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
etc.

IMPORTANT TERMS IN THE BINOMIAL EXPANSION

(a) General Term :

The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r} y^r$$

Ex.1 Find

(i) 28^{th} term of $(5x + 8y)^{30}$

(ii) 7^{th} term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

Sol. (i) $T_{27+1} = {}^{30}C_{27}\,(5x)^{30-27}\,(8y)^{27} = \frac{30!}{3!27!}\,(5x)^3\,.\,(8y)^{27}$

(ii) 7th term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

$T_{6+1} = {}^9C_6\left(\frac{4x}{5}\right)^{9-6}\left(-\frac{5}{2x}\right)^6 = \frac{9!}{3!6!}\left(\frac{4x}{5}\right)^3\left(\frac{5}{2x}\right)^6 = \frac{10500}{x^3}$

(b) Middle Term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by

$$T_{\frac{(n+2)}{2}} = C^n_{\frac{n}{2}}.x^{\frac{n}{2}}.y^{\frac{n}{2}}$$

(ii) If n is odd, there are two middle terms which are

$$T_{\frac{(n+1)}{2}} \;\&\; T_{\frac{(n+1)}{2}+1}$$

Middle term has greatest Binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$\Rightarrow$ ${}^{n}C_{r}$ will be maximum

When $r = \frac{n}{2}$ if n is even

When $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd

$\Rightarrow$ The term containing greatest Binomial coefficient will be middle term in the expansion of $(1 + x)^{n}$

Ex.2 Find the middle term(s) in the expansion of

(i) $(1 - \frac{x^2}{2})^{14}$ (ii) $(3a - \frac{a^3}{6})^{9}$

Sol.

(i) $(1 - \frac{x^2}{2})^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{th}$ term.

It means T_8 is middle term

$$T_s = C_7^{14}(-\frac{x^2}{2})^{9} = -\frac{429}{16}x^{14}$$

(ii) $(3a - \frac{a^3}{6})^{9}$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{th}$ & $\left(\frac{9+1}{2}+1\right)^{th}$

It means T_5 & T_6 is middle terms

$$T_5 = C_4^{9}(3a)^{9-4}\left(-\frac{a^3}{6}\right)^{4} = \frac{189}{8}a^{17}$$

$$T_6 = C_5^{9}(3a)^{9-5}(-\frac{a^3}{6})^{5} = -\frac{21}{16}a^{19}$$

(c) Term Independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Ex.3 Find the term independent of x in $[\sqrt{\frac{x}{3}} + \sqrt{(\frac{3}{2x^2})}]^{10}$.

Sol. General term in the expansion is

$$C_r^{10}\,(\frac{x}{3})^{\frac{r}{2}}(\frac{3}{2x^2})^{\frac{10-r}{2}} = C_r^{10}\,x^{\frac{3r}{2}-10}\cdot\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$

For constant term, $\frac{3r}{2} = 10 \quad \Rightarrow \quad r = \frac{20}{3}$

which is not an integer. Therefore, there will be no constant term.

(d) Numerically Greatest Term :

Binomial expansion of $(a + b)^n$ is as follows : –

$(a+b)^n$

$= {}^nC_0\,a^nb^0 + {}^nC_1\,a^{n-1}\,b^1 + {}^nC_2\,a^{n-2}\,b^2 + \ldots + {}^nC_r\,a^{n-r}\,b^r + \ldots\ldots + {}^nC_n\,a^0b^n$

If we put certain values of a and b in RHS, then each term of Binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r + 1)^{th}$ terms respectively

$T_r \quad = {}^nC_{r-1}\,a^{n-(r-1)}\,b^{r-1}$

$T_{r+1} \quad = {}^nC_r\,a^{n-r}\,b^r$

Now, $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}}\frac{a^{n-r}b^r}{a^{n-r+1}b^{r-1}}\right| = \frac{n-r+1}{r}.\left|\frac{b}{a}\right|$

Consider $\left|\frac{T_{r+1}}{T_r}\right| \geq 1$

$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \geq 1$

$\frac{n+1}{r} - 1 \geq \left|\frac{a}{b}\right|$

$r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Case-I

When $\frac{n+1}{1+|\frac{a}{b}|}$ is an integer (say m), then

(i) $T_{r+1} > T_r$ when $r < m$ $(r = 1, 2, 3 \ldots., m - 1)$

i.e. $T_2 > T_1$, $T_3 > T_2$, ……., $T_m > T_{m-1}$

(ii) $T_{r+1} = T_r$ when $r = m$

i.e. $T_{m+1} = T_m$

(iii) $T_{r+1} < T_r$ when $r > m$ ($r = m + 1, m + 2, ...n$)

i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ...$T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+|\frac{a}{b}|}$ is an integer, say m, then T_m and T_{m+1} will be numerically greatest terms (both terms are equal in magnitude)

Case - II

When $\frac{n+1}{1+|\frac{a}{b}|}$ is not an integer (Let its integral part be m), then

(i) $T_{r+1} > T_r$ when $r < \frac{n+1}{1+|\frac{a}{b}|}$ ($r = 1, 2, 3, ..., m-1, m$)

i.e. $T_2 > T_1$, $T_3 > T_2$, ..., $T_{m+1} > T_m$

(ii) $T_{r+1} < T_r$ when $r > \frac{n+1}{1+|\frac{a}{b}|}$ ($r = m + 1, m + 2, ...n$)

i.e. $T_{m+2} < T_{m+1}$, $T_{m+3} < T_{m+2}$, ..., $T_{n+1} < T_n$

Conclusion

When $\frac{n+1}{1+|\frac{a}{b}|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Notes:

(i) In any Binomial expansion, the middle term(s) has greatest Binomial coefficient.

In the expansion of $(a + b)^n$

n	No. of Greatest Binomial Coefficient	Greatest Binomial Coefficient
Even	1	${}^nC_{\frac{n}{2}}$
Odd	2	${}^nC_{\frac{(n-1)}{2}}$ and ${}^nC_{\frac{(n+1)}{2}}$ (Values of both these coefficients are equal)

(ii) In order to obtain the term having numerically greatest coefficient, put a = b = 1, and proceed as discussed above.

Ex.4 Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when x = ?

Sol. Using $\frac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \leq r \leq \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$\frac{11+1}{1+\left|\frac{3}{-5x}\right|} - 1 \leq r \leq \frac{11+1}{1+\left|\frac{3}{-5x}\right|}$

Solving we get $2 \leq r \leq 3$

r = 2, 3

So, the greatest terms are T_{2+1} and T_{3+1}.

Greatest term (when r = 2)

$T_3 = {}^{11}C_2 . 3^9 (-5x)^2 = 55.3^9 = T_4$

From above we say that the value of both greatest terms are equal.

Ex.5 If n is positive integer, then prove that the integral part of $(7 + 4)^n$ is an odd number.

Sol. Let $(7 + 4\sqrt{3})^n = I + f$ ....(i)

where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

Now $0 < 7 - 4\sqrt{3} < 1$

$0 < (7 - 4\sqrt{3})^n < 1$

Let, $(7 - 4\sqrt{3})^n = f'$ ....(ii)

$0<f<1$

Adding (i) and (ii)

$I + f + f' = (7+4\sqrt{3})^n + (7-4\sqrt{3})^n$

$= 2 [{}^nC_0 7^n + {}^nC_0 7^{n-2} (4\sqrt{3})^2 + ...]$

$I + f + f'$ = even integer $\Rightarrow$ ($f + f'$ must be an integer)

$0 < f + f' < 2 \Rightarrow f + f' = 1$

I+1=even integer therefore I is an odd integer.

Ex.6 What is the remainder when 5^{99} is divided by 13.

Sol. $5^{99} = 5.5^{98} = 5.(25)^{49} = 5(26-1)^{49}$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + \dots\dots + {}^{49}C_{48}\,(26)^1 - {}^{49}C_{49}\,(26)^0]$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + \dots\dots + {}^{49}C_{48}\,(26)^1 - 1]$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1(26)^{48} + \dots\dots + {}^{49}C_{48}\,(26)^1 - 13] + 60$

$= 13\,(k) + 52 + 8$ (where k is a positive integer)

$= 13\,(k+4) + 8$

Hence, remainder is 8.

Some Standard Expansions

(i) Consider the expansion

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, y^r$$

$$= {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + \dots\dots + {}^nC_r\, x^{n-r} y^r + \dots\dots + {}^nC_n\, x^0 y^n \quad \dots(i)$$

(ii) Now replace $y \to -y$ we get

$$(x-y)^n = \sum_{r=0}^{n} {}^nC_r\,(-1)^r\, x^{n-r}\, y^r$$

$$= {}^nC_0\, x^n y^0 - {}^nC_1\, x^{n-1} y^1 + \dots + {}^nC_r(-1)^r\, x^{n-r} y^r + \dots + {}^nC_n\,(-1)^n x^0 y^n \quad \dots(ii)$$

(iii) Adding (i) & (ii), we get

$$(x+y)^n + (x-y)^n = 2[{}^nC_0\, x^n y^0 + {}^nC_2\, x^{n-2}\, y^2 + \dots\dots]$$

(iv) Subtracting (ii) from (i), we get

$$(x+y)^n - (x-y)^n = 2[{}^nC_1\, x^{n-1} y^1 + {}^nC_3\, x^{n-3}\, y^3 + \dots\dots]$$

PROPERTIES OF BINOMIAL COEFFICIENTS

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots\dots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r \;;\; n \in N \quad \dots(i)$$

where $C_0, C_1, C_2, \dots\dots C_n$ are called combinatorial (Binomial) coefficients.

(a) The sum of all the Binomial coefficients is 2^n.

Put x = 1, in (i) we get

$$C_0 + C_1 + C_2 + \text{............} + C_n = 2^n \quad \Rightarrow \quad \sum_{r=0}^{n} {}^nC_r = 0 \qquad \text{....(ii)}$$

(b) Put x= –1 in (i) we get

$$C_0 - C_1 + C_2 - C_3 \text{............} + C_n = 0 \quad \Rightarrow \quad \sum_{r=0}^{n} (-1)^{r\,n} C_r = 0 \qquad \text{....(iii)}$$

(c) The sum of the Binomial coefficients at odd position is equal to the sum of the Binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii), $C_0 + C_2 + C_4 + \text{...........} = C_1 + C_3 + C_5 + \text{.......} = 2^{n-1}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n - r + 1}{r}$

(f) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r}.\dfrac{n-1}{r-1}\,{}^{n-2}C_{r-2} = \text{.......} = \dfrac{n(n-1)(n-2)\text{....}(n-r+1)}{r(r-1)(r-2)\text{.....}1}$

(g) ${}^nC_r = \dfrac{r+1}{n+1}.{}^{n+1}C_{r+1}$

Ex.7 Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \text{........} + {}^{10}C_{10} = {}^{26}C_{11}$

Sol. LHS $= {}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \text{.............} + {}^{25}C_{10}$

$\Rightarrow \quad {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \text{.......} + {}^{25}C_{10}$

$\Rightarrow \quad {}^{12}C_{11} + {}^{12}C_{10} + \text{........} + {}^{25}C_{10}$

$\Rightarrow \quad {}^{13}C_{11} + {}^{13}C_{10} + \text{.........}\,{}^{25}C_{10}$

and so on.

LHS $= {}^{26}C_{11}$

Aliter

LHS = coefficient of x^{10} in $\{(1 + x)^{10} + (1 + x)^{11} + \text{..............}\ (1+ x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1 + x)^{10} \frac{\{1+x\}^{16}-1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\frac{[(1+x)^{26}-(1+x)^{10}]}{x}$

$\Rightarrow$ coefficient of x^{11} in $[(1+x)^{26}-(1+x)^{10}] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Ex.8 Prove that :

(i) $C_1 + 2C_2 + 3C_3 + \text{........} + nC_n = n \,.\, 2^{n-1}$

(ii) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Sol.

(i) L.H.S. $= \sum_{r=1}^{n} r.{}^{n}C_r = \sum_{r=1}^{n} r.\frac{n}{r}.\ {}^{n-1}C_{r-1}$

$\sum_{r=1}^{n} r.{}^{n-1}C_{r-1} = n.[{}^{n-1}C_0 + {}^{n-1}C_0 + \ldots.. + {}^{n-1}C_{n-1}] \quad = n \,.\, 2^{n-1}$

Aliter : (Using method of differentiation)

$(1+x)^n = {}^{n}C_0 + {}^{n}C_1x + {}^{n}C_2x^2 + \text{.......} + {}^{n}C_nx^n$ ....(A)

Differentiating (A), we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \text{.......} + n.C_nx^{n-1}.$

Put x = 1,

$C_1 + 2C_2 + 3C_3 + \cdots \ldots \ldots + n \cdot C_n = n \cdot 2^{n-1}$

(ii) L.H.S. $= \sum_{r=0}^{n} \frac{C_r}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} \frac{n+1}{r+1}\ {}^{n}C_r$

$= \frac{1}{n+1}\sum_{r=0}^{n}\ {}^{n+1}C_{r+1} = \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + \ldots. + {}^{n+1}C_{n+1}\right] = \frac{1}{n+1}\left[2^{n+1} - 1\right]$

Aliter : (Using method of integration)

Integrating (A), we get

$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \text{.......} + \frac{C_nx^{n+1}}{n+1}$ (where C is a constant)

Put x = 0, we get, $C = -\frac{1}{-n+1}$

$\frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \text{.......} + \frac{C_nx^{n+1}}{n+1}$

Put x = 1, we get

$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \text{.......}\frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$

Put x = –1, we get

$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \cdots \ldots = \frac{1}{n+1}$

Ex.9 Prove that $C_1 - C_3 + C_5 - \ldots\ldots = 2^{\frac{n}{2}} \sin. \frac{n\pi}{4}$

Sol. Consider the expansion $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots + C_n x^n$ ....(i)

putting x = – i in (i) we get

$(1 - i)^n = C_0 - C_1 i - C_2 + C_3 i + C_4 + \ldots\ldots. (-1)^n C_n i^n$

Or $2^{\frac{n}{2}}\left[\cos\left(-\frac{n\pi}{4}\right) + i\sin\left(-\frac{n\pi}{4}\right)\right]$

$= (C_0 - C_2 + C_4 - \cdots \ldots.) - i(C_1 - C_3 + C_5 - \cdots \ldots)$ ....(ii)

Equating the imaginary part in (ii) we get

$C_1 - C_3 + C_5 - \cdots \ldots \ldots = 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$

Ex.10 If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots.. + C_n x^n$ then prove that

$$\sum_{0 \le i < j \le n} \left(C_i + C_j\right)^2 = (n-1)^{2n}C_n + 2^{2n}$$

Sol. L.H.S $\sum_{0 \le i < j \le n} \left(C_i + C_j\right)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + \ldots. + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \ldots.$

$+ (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \ldots + (C_2 + C_n)^2 + \ldots. + (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2) + 2 \sum_{0 \le i < j \le n} C_i C_j$

$= n^{2n}C_n + 2 \cdot \left\{2^{2n-1} - \frac{2n!}{2 \cdot n! n!}\right\}$

$n. {}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1). {}^{2n}C_n + 2^{2n} = \text{R.H.S.}$