BINOMIAL THEOREM

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BINOMIAL THEOREM FOR POSITIVE INDEX

Such formula by which any power of a binomial expression can be expanded in the form of a series is known as Binomial Theorem. For a positive integer n , the expansion is given by

$$(a+x)^n = {}^nC_0a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \dots + {}^nC_ra^{n-r}x^r + \dots + {}^nC_nx^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}x^r$$

where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, . . . , ${}^nC_n$ are called Binomial co-efficients. Similarly

$$(a-x)^n = {}^nC_0a^n - {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 - \dots + (-1)^r\,{}^nC_ra^{n-r}x^r + \dots + (-1)^n\,{}^nC_nx^n$$

i.e. $(a-x)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_ra^{n-r}x^r$

Replacing a = 1, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_rx^r + \dots + {}^nC_nx^n$$

and $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - \dots + (-1)^r\,{}^nC_rx^r + \dots + (-1)^n\,{}^nC_nx^n$

Observations:

- There are (n+1) terms in the expansion of $(a+x)^n$.
- Sum of powers of x and a in each term in the expansion of $(a+x)^n$ is constant and equal to n.
- The general term in the expansion of $(a+x)^n$ is $(r+1)^{th}$ term given as $T_{r+1} = {}^nC_r\,a^{n-r}x^r$
- The p^{th} term from the end = $(n-p+2)^{th}$ term from the beginning .
- Coefficient of x^r in expansion of $(a+x)^n$ is ${}^nC_r\,a^{n-r}x^r$.
- ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$.
- In the expansion of $(a+x)^n$ and $(a-x)^n$, x^r occurs in $(r+1)^{th}$ term.

Ex.1 If the coefficients of the second, third and fourth terms in the expansion of $(1 + x)^n$ are in A.P., show that n = 7.

Sol. According to the question $^nC_1 \cdot {}^nC_2 \cdot {}^nC_3$ are in A.P.

$$\frac{2n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$

$n^2 - 9n + 14 = 0$

$\Rightarrow \quad (n - 2)(n - 7) = 0$

$\Rightarrow \quad n = 2$ or 7

Since the symbol $^nC_3$ demands that n should be ≥ 3

n cannot be 2, $\therefore$ n = 7 only.

Ex.2 Find the

(i) last digit

(ii) last two digit

(iii) last three digit of 17^{256}.

Sol. $17^{256} = 289^{128} = (290 - 1)^{128}$

$= {}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + \ldots\ldots\ldots + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$

$= 1000m + {}^{128}C_2(290)^2 - {}^{128}C_1(290) + 1$

$= 1000m + \frac{128 \times 127}{2} \times (290)^2 - \frac{128 \times 290}{1} + 1 = 1000m + 683527680 + 1$

Hence the last digit is 1. Last two digits is 81. Last three digit is 681.

Ex.3 If the binomial coefficients of $(2r + 4)^{th}$, $(r - 2)^{th}$ term in the expansion of $(a + bx)^{18}$ are equal find r.

Sol. This is possible only when

either $2r + 3 = r - 3$ ....(1)

or $2r + 3 + r - 3 = 18$ ....(2)

from (1) $r = -6$ not possible but from (2) $r = 6$

Hence r = 6 is the only solution.

Ex.4 Find the coefficient of

(i) x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$,

(ii) and x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$.

Find the relation between a and b if these coefficients are equal.

Sol. The general term in $\left(ax^2 + \frac{1}{bx}\right)^{11} = {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r$

$$= {}^{11}C_r \frac{a^{11-r}}{b^r} x^{22-3r}$$

If in this term power of x is 7, then 22 – 3r = 7 ⇒ r = 5

∴ coefficient of x7 = ${}^{11}C_5 \frac{a^6}{b^5}$ ...(1)

The general term in $\left(ax - \frac{1}{bx^2}\right)^{11} = (-1)^r \, {}^{11}C_r (ax)^{11-r} \left(\frac{1}{bx^2}\right)^r$

$$= (-1)^r \, {}^{11}C_r \frac{a^{11-r}}{b^r} x^{11-3r}$$

If in this term power of x is –7, then 11 – 3r = –7 ⇒ r = 6

∴ coefficient of $x^{-7} = (-1)^6 \, {}^{11}C_6 \frac{a^{11-6}}{b^6} = {}^{11}C_5 \frac{a^5}{b^6}$

If these two coefficient are equal, then ${}^{11}C_5 \frac{a^6}{b^5} = {}^{11}C_5 \frac{a^5}{b^6}$

$a^6 b^6 = a^5 b^5$

$\Rightarrow a^5 b^5 (ab - 1) = 0$

$\Rightarrow ab = 1 (a \neq 0, \ b \neq 0)$