PERMUTATIONS AND COMBINATIONS

PERMUTATIONS

Permutation & combination

(a) Factorial:

A Useful Notation: $n! = n.(n-1).(n-2)\ldots\ldots\ldots\ldots.3.\,2.\,1;\ n! = n.(n-1)!$ Where $n \in N$

Notes:

1. $0! = 1! = 1$

2. Factorials of negative integers are not defined.

3. n! is also denoted by

4. $(2n)! = 2^n.n!\,[1.\,3.\,5.\,7\ldots\ldots..(2n-1)]$

5. Prime factorization of n! : Let p be a prime number and n be a positive integer, then exponent of p in n! is denoted by E_p (n!) and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots..+\left[\frac{n}{p^k}\right]$$

Where $p^k \le n < p^{k+1}$ and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number n, then n can be written as

$$n = 2^{\alpha_i} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4}$$

Where a_i are whole numbers.

(b) Permutation:

Each of the arrangements in a definite order which can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained.

Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

${}^nP_r$ denotes the number of permutations of n different things, taken r at a time

$(n \in N, r \in W, r \leq n)$

$$P_r^{\ n} = n(n-1)(n-2) \ldots\ldots\ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}$$

- ${}^{n}P_{n} = n!,\quad {}^{n}P_{0} = 1,\quad {}^{n}P_{1} = n$
- Number of arrangements of n distinct things taken all at a time = n!

${}^{n}P_{r}$ is also denoted by A_r^n or P(n,r).

(c) Combination:

Each of the groups or selections which can be made by taking some or all of the things without considering the order of the things in each group is called a COMBINATION. Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

${}^{n}C_{r}$ denotes the number of combinations of n different things taken r at a time

$$(n \in N, r \in W, r \leq n)$$

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Note:

1. ${}^{n}C_{r}$ is also denoted by or C (n, r).

2. ${}^{n}P_{r} = {}^{n}C_{r}.\, r!$

Ex. How many three digit can be formed using the digits 1, 2, 3, 4, 5, without repetition of digits? How many of these are even?

Sol. Three places are to be filled with 5 different objects.

Number of ways $= {}^{5}P_{3} = 5 \times 4 \times 3 = 60$

For the 2nd part, unit digit can be filled in two ways & the remaining two digits can be filled in ${}^{4}P_{2}$ ways.

Number of even numbers $= 2 \times {}^{4}P_{2} = 24$.

Ex. Find the exponent of 6 in 50!

$E_2(50!) = [\frac{50}{2}] + [\frac{50}{4}] + [\frac{50}{8}] + [\frac{50}{16}] + [\frac{50}{32}] + [\frac{50}{64}]$

Sol. (where [ ] denotes integral part)

$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$

$E_3(50!) = [\frac{50}{3}] + [\frac{50}{9}] + [\frac{50}{27}] + [\frac{50}{81}]$

$E_3(50!) = 16 + 5 + 1 + 0 = 22$

50! can be written as $50! = 2^{47}.\,3^{22}$.........

Therefore exponent of 6 in 50! = 22

Ex. If a denotes the number of permutations of (x + 2) things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of (x – 11) things taken all at a time such that a = 182 bc, then the value of x is

Sol. ${}^{x+2}P_{x+2} = a \quad \Rightarrow \quad a = (x+2)!$

${}^{x}P_{11} = b \quad \Rightarrow \quad b = \frac{x!}{(x-11)!}$

${}^{x-11}P_{x-11} = c \quad \Rightarrow \quad c = (x-11)\,!$

$a = 182bc$

$(x+2)! = 182\,\frac{x!}{(x-11)!}\,(x-11)! \quad \Rightarrow \quad (x+2)(x+1) = 182 = 14 \times 13$

$x + 1 = 13 \quad \Rightarrow \quad x = 12$

Ex. How many 4 letter words can be formed from the letters of the word ‘ANSWER’? How many of these words start

with a vowel ?

Sol. Number of ways of arranging 4 different letters from 6 different letters are .

${}^{6}C_4 4! = \frac{6!}{2!} = 360$

There are two vowels (A & E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A .... = ${}^{5}C_3 3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E .... = ${}^{5}C_3 3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with a vowel = 60 + 60 = 120.

PROPERTIES OF $^{n}p_{r}$ and $^{n}c_{r}$

1. The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!.^{n-p}C_{r-p}$ $(p \le r \le n)$

2. The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^{r}.

3. Following properties of $^{n}C_{r}$ should be remembered :

1) $^{n}C_{r} = {}^{n}C_{n-r}$; $^{n}C_{0} = {}^{n}C_{n} = 1$

2) $^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $x + y = n$

3) $^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

4) $^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \ldots\ldots\ldots\ldots + {}^{n}C_{n} = 2^{n}$

5) $^{n}C_{r} = {}^{n-1}C_{r-1}$

6) $^{n}C_{r}$ is maximum when $r = \frac{n}{2}$ if n is even & $r = \frac{n-1}{2}$ or $r = \frac{n+1}{2}$ if n is odd.

7) The number of combinations of n different things taking r at a time,

(i) When p particular things are always to be included = $^{n-p}C_{r-p}$

(ii) When p particular things are always to be excluded = $^{n-p}C_{r}$

(iii) when p particular things are always to be included and q particular things are to be excluded = $^{n-p-q}C_{r-p}$

Ex. If $^{49}C_{3r-2} = {}^{49}C_{2r+1}$, find 'r'.

Sol. $^{n}C_{r} = {}^{n}C_{s}$ if either $r = s$ or $r + s = n$.

Thus $3r - 2 = 2r + 1 \Rightarrow r = 3$

$3r - 2 + 2r + 1 = 49 \Rightarrow 5r - 1 = 49 \Rightarrow r = 10$

$\therefore$ $r = 3, 10$

Ex. If all the letters of the word 'QUEST' are arranged in all possible ways and put in dictionary order, then find the rank of the given word.

Sol. Number of words beginning with E $= {}^4P_4 = 24$

Number of words beginning with QE $= {}^3P_3 = 6$

Number of words beginning with QS $= 6$

Number of words beginning with QT $= 6$.

Next word is 'QUEST'

its rank is $24 + 6 + 6 + 6 + 1 = 43$.

Ex. There are three coplanar parallel lines. If any p points are taken on each of the lines, then find the maximum number of triangles with vertices at these points.

Sol. The number of triangles with vertices on different lines $= {}^pC_1 \times {}^pC_1 \times {}^pC_1 = p^3$

The number of triangles with two vertices on one line and the third vertex on any one of the other two lines

$$ {}^3C_1\{{}^pC_2 \times {}^{2P}C_1\} = 6p $$

$$ \frac{p(p-1)}{2} $$

So, the required number of triangles $= p^3 + 3p^2 (p - 1) = p^2 (4p - 3)$

Ex. There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive ?

Sol. Total number of remaining non-selected points $= 6$

. . . . . .

Total number of gaps made by these 6 points $= 6 + 1 = 7$

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x . . x . x . . x .

Total number of ways of selecting 4 gaps out of 7 gaps $= {}^7C_4$

Formation of groups

1. **(i)** The number of ways in which $(m+n)$ different things can be divided into two groups such That one of them contains m things and other has n things, is

$$\frac{(m+n)!}{m!n!}\ (m \neq n).$$

(ii) If $m = n$, it means the groups are equal & in this case the number of divisions Is $\frac{(2n)!}{n!n!2!}$. As in any one way it is possible to interchange the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways:

$$\frac{(2n)!}{n!\,n!\,2!}$$

2. **(i)** Number of ways in which $(m+n+p)$ different things can be divided into three groups containing m, n & p things respectively is : , $\frac{(m+n+p)!}{m!n!p!}$ $m \neq n \neq p$....

(ii) If $m = n = p$ then the number of groups $= . = \frac{(3n)!}{n!n!n!3!}$

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is. $\frac{(3n)!}{(n!)^3}$

3. In general, the number of ways of dividing n distinct objects into l groups containing p objects each and m groups containing q objects each is equal to $\frac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ell!m!}$

Here $lp + mq = n$

Ex. 12 different toys are to be distributed to three children equally. In how many ways this can be done?

Sol. The problem is to divide 12 different things into three different groups.

Number of ways $= \frac{12!}{4!4!4!} = 34650$

Ex. In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together ? Also find the number of ways if these groups are to be sent to three different colleges.

Sol. Assuming two particular students as one student (as they are always together), we have to make groups of $5 + 5 + 4$ students out of 14 students.

Therefore total number of ways $= = \frac{14!}{5!5!4!2!}$

Now if these groups are to be sent to three different colleges, the total number of ways

$$= \frac{14!}{5!5!4!2!} \times 3!$$

Ex. Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

Sol. Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups

$$= \frac{48!}{(12!)^4 4!}$$

Now, distribute exactly one Ace to each group of 12 cards. Total number of ways =

$$= \frac{48!}{(12!)^4 4!} \times 4!$$

Now, distribute these groups of cards among four players

$$= \frac{48!}{(12!)^4 4!} \times 4!\, 4! = \frac{48!}{(12!)^4} \times 4!$$

Circular permutation

The number of circular permutations of n different things taken all at a time is (n - 1)!

If clockwise & anti-clockwise circular permutations are considered to be same,then it is.

Ex. In how many ways can we arrange 6 different flowers in a circle? In how many ways we can form a garland using these flowers?

Sol. The number of circular arrangements of 6 different flowers = (6 – 1)! = 120

When we form a garland, clockwise and anticlockwise arrangements are similar. Therefore, the
number of ways of forming garland = $(6 - 1)! = 60$.

Ex. In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

Sol. Leaving one seat vacant between two boys, 5 boys may be seated in 4! Ways. Then at remaining 5 seats, 5 girls sit in 5! Ways. Hence the required number of ways = $4! \times 5$

Ex. A person invites a group of 10 friends at dinner. They sit

(i) 5 on one round table and 5 on other round table,

(ii) 4 on one round table and 6 on other round table.

Find the number of ways in each case in which he can arrange the guests.

Sol. **(i)** The number of ways of selection of 5 friends for first table is ${}^{10}C_5$. Remaining 5 friends are left for second table.

The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is ${}^{10}C_5 \times 4! \times 4! =$

(ii) The number of ways of selection of 6 guests is ${}^{10}C_6$.

The number of ways of permutations of 6 guests on round table is 5!. The number of permutations of 4 guests on round table is 3!

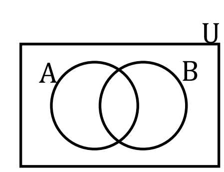
Therefore, total number of arrangements is :

Principle of Inclusion and Exclusion

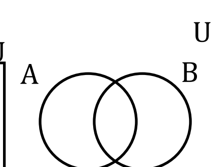
In the Venn's diagram **(i)**, we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A' \cap B') = n(U) - n(A \cup B)$



(i)



(ii)

In the Venn's diagram **(ii)**, we get

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

In general, we have n $(A_1 \cup A_2 \cup \ldots\ldots\cup A_n)$

Ex. Find the number of permutations of letters a,b,c,d,e,f,g taken all at a time if neither 'beg' nor 'cad' pattern appear.

Sol. The total number of permutations without any restrictions; $n(U) = 7!$

Let A be the set of all possible permutations in which 'beg' pattern always appears :

$$n(A) = 5!$$

Let B be the set of all possible permutations in which 'cad' pattern always appears:

$$n(B) = 5!$$

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear.

$n(A \cap B) = 3!$. Therefore, the total number of permutations in which 'beg' and 'cad' patterns do not appear

$$n(A' \cap B') = n(U) - n(A\cap B) = n(U) - n(A) - n(B) + n(A\cap B)$$
$$= 7! - 5! - 5! + 3!.$$

Arrangement of n Things, Those are not All Different

The number of permutations of 'n' things, taken all at a time, when 'p' of them Are same & of one type, q of them are same & of second type, 'r' of them are Same & of a third type & the remaining

n - (p + q + r) things are all different, is.

Ex. In how many ways we can arrange 3 red flowers, 4 yellow flowers and 5 white flowers in a row? In how many ways this is possible if the white flowers are to be separated in any arrangement?

(Flowers of same color are identical).

Sol. Total we have 12 flowers 3 red, 4 yellow and 5 white.

Number of arrangements = = 27720.

For the second part, first arrange 3 red & 4 yellow

This can be done in = 35 ways

Now select 5 places from among 8 places (including extremes) & put the white flowers there.

This can be done in ${}^{8}C_{5} = 56$.

$\therefore$ The number of ways for the 2^{nd} part $= 35 \times 56 = 1960$.

Ex. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the
odd places?

Sol. There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh). At these places the odd digits can be arranged in ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in ways

$\therefore$ The required number of numbers $= 6 \times 3 = 18$.

Ex. Find the total number of 4 letter words formed using four letters from the word ''PARALLELOPIPED'.

Sol. Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No of ways of selection	No. of ways of arrangement	Total
All district	${}^{8}C_{4}$	${}^{8}C_{4} \times 4!$	1680
2 a like, 2 district	$4C_{1} \times {}^{7}C_{2}$	$4C1 \times {}^{7}C_{2}$	1008
2 a like, 2 other a like	${}^{4}C_{2}$	$4C_{2} \times \frac{4!}{2!\,2!}$	36
3 a like, a district	${}^{2}C_{1} \times {}^{7}C_{1}$	${}^{2}C_{1} \times {}^{7}C_{1} \times \frac{4!}{3!}$	56
		Total	2780