PERMUTATIONS AND COMBINATIONS

COMBINATION

Combination

If ${}^{n}C_{r}$ denotes the number of combinations (selections) of n different things

Taken r at a time, then

$$n_{C_1} = \frac{n!}{r!(n-r)!} = \frac{n_{P_r}}{r!}$$

Where

$r \leq n$; $n \in N$ and $r \in W$.

A. Given n different objects, the number of ways of selecting at least one of them is,

$${}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \ldots\ldots + {}^{n}C_n = 2^n - 1.$$

This can also be stated as the total number of combinations of n distinct things.

B.

(i) Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \ldots\ldots$things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1)\ldots\ldots\ldots -1$.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by : $(p + 1)(q + 1)(r + 1)\, 2^n - 1$

Note:

(i) ${}^{n}C_r = {}^{n}C_{n-r}$

(ii) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

(iii) ${}^{n}C_r = 0$ if r $\{0, 1, 2, 3\ldots\ldots\ldots, n\}$

Ex. Fifteen players are selected for a cricket match.

(i) In how many ways the playing 11 can be selected?

(ii) In how many ways the playing 11 can be selected including a Particular player?

(iii) In how many ways the playing 11 can be selected excluding two Particular players?

Sol. (i) 11 players are to be selected from 15

Number of ways = ${}^{15}C_{11} = 1365$.

(ii) Since one player is already included, we have to select 10 from the remaining 14

Number of ways = ${}^{14}C_{10} = 1001$.

(iii) Since two players are to be excluded, we have to select 11 from the Remaining 13.

Number of ways = ${}^{13}C_{11} = 78$.

Ex. There are 3 books of Mathematics, 4 of Science and 5 of English. How many different collections can be made such that each collection consists of-?

(i) One book of each subject?

(ii) At least one book of each subject?

(iii) At least one book of English?

Sol. (i) ${}^{3}C_{1} \times {}^{4}C_{1} \times {}^{5}C_{1} = 60$

(ii) $(2^3-1)\ (2^4-1)\ (2^5-1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5-1)\ (2^3)\ (2^4) = 31 \times 128 = 3968$

DIVISORS

Let $N = p^a . q^b . r^c$ .... where p, q, r........ are distinct primes & a, b, c....... are natural numbers then:

(a) The total numbers of divisors of N including 1 & N is $= (a + 1)\ (b + 1)\ (c + 1)$.......

(b) The sum of these divisors is =

$(p^0 + p^1 + p^2 + ... + p^a)\ (q^0 + q^1 + q^2 + ... + q^b)\ (r^0 + r^1 + r^2 + ... + r^c)$...

(c) Number of ways in which N can be resolved as a product of two factor is

$\frac{1}{2}(a + 1)(b + 1)(c + 1) \ldots \ldots$ if N is not a perfect square

$\frac{1}{2}[(a + 1)(b + 1)(c + 1) \ldots \ldots + 1]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Note:

(i) Every natural number except 1 has at least 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.

(ii) A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.

(iii) Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, **Eg.** 4 and 25.

(iv) 1 is neither prime nor composite however it is co-prime with every other natural number.

(v) Two prime numbers are said to be twin prime numbers if their non-negative difference is 2

(E.g.5 & 7, 19 & 17 etc).

(vi) All divisors except 1 and the number itself are called proper divisors.

Ex. Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Sol. (i) The number $38808 = 2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

$= (3 + 1) (2 + 1) (2 + 1) (1 + 1) – 2 = 70$

(ii) The sum of these divisors

$=(2^0 + 2^1 + 2^2 + 2^3) (3^0 + 3^1 + 3^2) (7^0 + 7^1 + 7^2) (11^0 + 11^1) – 1 – 38808$

$= (15) (13) (57) (12) – 1 – 38808 = 133380 – 1 – 38808 = 94571.$

Ex. In how many ways the number 18900 can be split in two factors which are relative prime (or coprime) ?

Sol. Here N = 18900 $= 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in 18900 = n = 4

Hence number of ways in which 18900 can be resolved into two factors which are relative prime

(or coprime) $= 2^{4-1} = 2^3 = 8$.

Total distribution

(a) Distribution of Distinct Objects:

Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by: p^n

(b) Distribution of Alike Objects :

Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Ex. Find the number of solutions of the equation $x + y + z = 6$, where $x, y, z \in W$.

Sol. Number of solutions = coefficient of x^6 in $(1 + x + x^2 + ... x^6)^3$

$= \text{coefficient of } x^6 \text{ in } (1 - x^7)^3 (1 - x)^{-3}$

$= \text{coefficient of } x^6 \text{ in } (1 - x)^{-3}$

$= {}^{3+6-1}C_6 = {}^8C_2 = 28.$

Ex. In how many ways can 5 different mangoes, 4 different oranges & 3 different apples be distributed among 3 children such that each gets alteast one mango ?

Sol. 5 different mangoes can be distributed by following ways among 3 children such that each gets at least 1:

3 1 1

2 2 1

Total number of ways: $\left(\frac{5!}{3!1!1!2!} + \frac{5!}{2!2!2!}\right) \times 3!$

Now, the number of ways of distributing remaining fruits (i.e. 4 oranges + 3 apples) Among 3 children = 3^7 (as each fruit has 3 options).

$\therefore$ Total number of ways $= \left(\frac{5!}{3!2!} + \frac{5!}{(2!)^3}\right) \times 3! \times 3^7$

Ex. Find the number of non negative integral solutions of the inequation $x + y + z \leq 20$.

Sol. Let w be any number $(0 \leq w \leq 20)$, then we can write the equation as :

$x + y + z + w = 20$ (here $x, y, z, w \geq 0$)

Total ways = ${}^{23}C_3$

Arrangements

If ${}^nP_r$ denotes the number of permutations (arrangements) of n different things, taking r at a time,

then

$${}^nP_r = n (n - 1) (n - 2)..... (n - r + 1) = \frac{n!}{(n-r)!}$$

Dearrangement

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes
(one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\,[1 - \frac{1}{1!} + \frac{1}{2!} + \cdots .. + \frac{(-1)^n}{n!}]$$

Proof:

n letters are denoted by 1,2,3,........,n. Let A_i denote the set of distribution of letters in Envelopes (one letter in each Envelope) so that the i^{th} letter is placed in the corresponding Envelope. Then, $n(A_i) = 1 \times (n-1)!$ [Since the remaining n–1 letters can be placed in n –1 Envelops in (n–1)! ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes.

Then,

$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$

Also $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$n(A_1' \cup A_2' \cup ..... \cup A_n') = n! - n(A_1 \cup A_2 \cup .... \cup A_n)$

$= n! - [\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + \cdots ... + (-1)^n \sum n(A_i \cap A_2 ... \cap\cap A_n)]$

$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + ...+ (-1)^{n-1} \times {}^nC_n 1]$

$$= n! - [\frac{n!}{1!\,(n-1)!}(n-1)! - \frac{n!}{2!\,(n-2)!}(n-2)! + \cdots ... + (-1)^{n-1}]$$

$$= n!\,[1 - \frac{1}{1!} + \frac{1}{2!} + \cdots ... ... + \frac{(-1)^n}{n!}]$$

Ex. A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that

(i) All the letters are in the wrong envelopes.

(ii) At least two of them are in the wrong envelopes.

Sol. (i) The number of ways is which all letters be placed in wrong envelopes

$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$

$= 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$

$= 360 - 120 + 30 - 6 + 1 = 265.$

(ii) The number of ways in which at least two of them in the wrong envelopes

$= {}^6C_4 . 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 . 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) +$

$+ {}^6C_2 . 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) + {}^6C_1 . 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$+ {}^6C_0\ 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$

$= 15 + 40 + 135 + 264 + 265 = 719$