COMPLEX NUMBERS AND QUADRATIC EQUATIONS

INTRODUCTION OF COMPLEX NUMBER

IMAGINARY NUMBERS

Square root of a negative number is called imaginary number. While solving the equations $x^2 + 1 = 0$, a quantity $\sqrt{-1}$ is obtaind and is denoted by i (iota) which is imaginary.

Further $\sqrt{-2}$ is an imaginary number and can be written as

$\sqrt{-2} = \sqrt{2} \times \sqrt{-1} = \sqrt{2}i$

If $a < 0$, then $\sqrt{a} = \sqrt{|a|}j$

Integral Powers of i

We have $i = \sqrt{-1}$ so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

For any $n \in N$, we have

$i^{4n+1} = i$, $i^{4n+2} = -1$

$i^{4n+3} = -i$, $i^{4n} = 1$

Thus any integral power of i can be expressed as ± 1 or $\pm i$.

In other words . $i^n \begin{cases} (-1)^{\frac{n}{2}} \text{ if n is even integer} \\ (-i)^{\frac{n-1}{2}} \text{; if n is odd integer} \end{cases}$.

Also $i^{-n} = \dfrac{1}{i^n}$

Ex.1 Evaluate:

(i) j^{786} (ii) $(-\sqrt{-1})^{23}$ (iii) $\dfrac{i^2 + j^3 + j^4 + j^5}{i + j^2 + i^3}$

Sol. (i) $i^{786} = i^{4\times196+2} = i^2 = -1$

(ii) $(-\sqrt{-1})^{23} = (-1\times i)^{23} = (-i)^{23} = -(i)^{23} = -(i)^{4\times5+3} = -i^3 = -(-i) = i$

(iii) $\dfrac{i^2+j^3+j^4+j^5}{i+j^2+i^3} = \dfrac{i^2\left(1+i+j^2+j^3\right)}{i+j^2+j^3} = \dfrac{(-1)(1+i-1-i)}{i-1-i} = \dfrac{0}{-1} = 0$

Note : $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$ is true iff at least one of a and b are non-negative. If $a < 0$ and $b < 0$, then

$$\sqrt{a}\times\sqrt{b} = \sqrt{-|a|}\times\sqrt{-|b|}$$

$$= i\sqrt{|a|}\times i\sqrt{|b|} = -\sqrt{ab}$$

Complex Numbers

The formal addition, 'a + ib', where a, b $\in$ R and the collection of all such expressions is called the set of complex numbers. For the complex number, $z = a + ib$, 'a' is called as real part of z and is denoted by Re(z) while 'b' is called as imaginary part of z and is denoted by Im(z).

Set of complex numbers is denoted by C, which includes the set of real numbers R.

i.e., $R \subset C$

A complex number z is said to be purely real if $I_m(z) = 0$ and is said to be purely imaginary if $Re(z) = 0$. The complex number $0 + 0i = 0$ is both purely real and purely imaginary.

All purely imaginary numbers except zero are imaginary numbers but an imaginary number may or may not be purely imaginary.

For e.g., $4 + 3i$ is imaginary but not purely imaginary.

Equality of Complex numbers

Two complex numbers $a + ib$ and $c + id$ are said to be equal, if and only if, $a = c$ and $b = d$. i.e., the corresponding real and imaginary parts are equal.

If $a + ib = C_1$ and $c + id = C_2$

then either $C_1 = C_2$ or $C_1 \neq C_2$

For imaginary numbers, the property of order is not defined because i is neither positive, zero nor negative.

So $C_1 > C_2$ or $C_2 > C_1$ is meaningless till b and d are both equal to zero.

i.e., $C_1 > C_2$ or $C_1 < C_2$ are meaningless if b and d are not equal to zero.

ALGEBRA OF COMPLEX NUMBERS

(i) Addition : $(a + ib) + (c + id) = (a + c) + i\,(b + d)$

(ii) Subtraction : $(a + ib) - (c + id) = (a - c) + i\,(b - d)$

(iii) Multiplication : $(a + ib) \cdot (c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc)$

(iv) Division : $\frac{a+ib}{c+id} = \frac{ac+bd}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$

(When at least one of c and d is non-zero)

Conjugate Complex Number

For $z = a + ib$, its conjugate is defined as $z = a - ib$. Here the complex conjugate is obtained just by changing sign of i.

Properties of conjugate :

(i) $\overline{(z)} = Z$

(ii) $z = \bar{Z}$ = iff z is purely real

(iii) $z + \bar{Z} = 2\text{Re}(z) \quad \Rightarrow \quad \text{Re}(z) = \text{Re}(\bar{z}) = \frac{z+\bar{z}}{2}$

(iv) $z - \bar{Z} = 2i\,I_m(z) \quad \Rightarrow \quad I_m(z) = \frac{z-\bar{z}}{2i}$

(v) $z = -\bar{Z}$ iff z is purely Imaginary

(vi) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

(vii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(viii) $\left(\frac{\overline{z_1}}{\overline{z_2}}\right) = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

(ix) $\left(\overline{z^{n}}\right) = (\overline{z})^{n}$

(x) $z_1\overline{z_2} + \overline{z_1}z_2 = 2\mathrm{Re}\left(\overline{z_1}z_2\right) = 2\mathrm{Re}\left(z_1\overline{z_2}\right)$

(xi) If $z = f(z_1)$, then $\overline{z} = f(\overline{z}_1)$

Modulus of a Complex Number

The modulus of a complex number $z = x + iy$ is defined as

$|z| = \sqrt{x^2 + y^2} = \sqrt{\{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2}$. In other way distance of a complex number z from origin while represented on argand plane is called as modulus of a complex number denoted by mod(z) or |z|, or r.

Here $OP = r = \sqrt{x^2 + y^2}$

|z| is also called absolute value of z.

Note : $|z_1 - z_2|$ is the distance between z_1 and z_2.

Properties of Modulus :

(i) $|z| \geq 0; |z| = 0$ iff real and imaginary parts are zero

(ii) $|z_1z_2| = |z_1||z_2|$. In general $|z_1z_2 \ldots\ldots z_n| = |z_1||z_2|\ldots|z_n|$

(iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, (z_2 \neq 0)$

(iv) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

(v) $\overline{z} = |z|^2$

(vi) $-|z| \leq \mathrm{Re}(z) \leq |z| \; -|z| \leq \mathrm{Im}(z) \leq |z|$

(vii) $|z^n| = |z|^n$

(viii) $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2)$

$$= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2$$

$$= |z_1|^2 + |z_2|^2 + 2\text{Re}\left(z_1\overline{z}_2\right)$$

(ix) $|z_1 - z_2|^2 = \left(z_1 - z_2\right)\left(\overline{z}_1 - \overline{z}_2\right)$

$$= z_1\overline{z}_1 - z_1\overline{z}_2 - z_2\overline{z}_1 + z_2\overline{z}_2$$

$$= |z_1|^2 + |z_2|^2 - 2\text{Re}\left(z_1\overline{z}_2\right)$$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

(xi) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \left(a^2 + b^2\right)\left(|z_1|^2 + |z_2|^2\right)$ where $a, b \in R$