COMPLEX NUMBERS AND QUADRATIC EQUATIONS

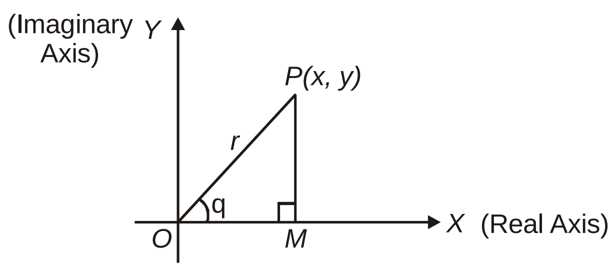
ARGAND PLANE AND POLAR REPRESENTATION

Representation of a Complex Number

Geometrical Representation

The complex number, $z = x + iy$ can be associated with the ordered pair $P(x, y)$.

We consider two perpendicular lines OX and OY (analogous to the Cartesian system) called as the real axis and imaginary axis respectively and where O denotes the origin of reference. The resulting plane is called as Argand plane or Gaussian plane or complex plane and z is represented by the point P corresponding to the ordered pair (x, y). The point P is called as affix of z.



Argument or Amplitude of z

The argument of z, denoted by arg z or amp z is the angle which OP makes with the positive direction of real axis, the angle being measured in anticlockwise sense.

If $z = x + iy$ then the angle q given by $\tan\theta = \frac{y}{x}$ is said to be the argument of z.

or $$\arg(z) = \text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\text{lm}(z)}{\text{Re}(z)}\right)$$

Note : Arg (z) is not unique; if q_1 is one value, then $\theta_1 + 2k\pi : k \in I$ is the set of all possible values of arg z. Any two arguments of a complex number differ by $2k\pi$.

Principal argument of a complex number z is that value of argument (z) which lies in the interval (–p, p].

For z = x + iy, $\theta = \tan^{-1}\left|\frac{y}{x}\right|$, then principal argument depends on the quadrant in which point (x, y) lies.

(i) x > 0, y = 0 ⇒ Principal argument = 0 (Positive Real Axis)

(ii) x > 0, y > 0 ⇒ Prinicpal argument = q (I^{st} quadrant)

(iii) x = 0, y > 0 ⇒ Prinicpal argument = $\frac{\pi}{2}$ (Positive imaginary axis)

(iv) x < 0, y > 0 ⇒ Principal argument = p – q (II^{nd} quadrant)

(v) x < 0, y = 0 ⇒ Prinicpal argument = p (Negative Real Axis)

(vi) x < 0, y < 0 ⇒ Principal argument = –p + q (III^{rd} quadrant)

(vii) x = 0, y < 0 ⇒ Principal argument = $-\frac{\pi}{2}$ (Negative Imaginary Axis)

(viii) x > 0, y < 0 ⇒ Principal argument = –θ (iv^{th} quadrant)

Properties of Argument

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi : k = 0, \pm 1$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi : k = 0, \pm 1$

(iii) $\arg(z^2) = 2\arg(z) + 2k\pi : k = 0, \pm 1$

(iv) If arg(z) = 0 or π then z is real

(v) $\arg\left(\frac{z}{\overline{z}}\right) = 2\arg(z) + 2k\pi : \quad k = 0, \pm 1$

(vi) $\arg(z^n) = n\arg(z) + 2k\pi : k = 0, \pm 1$

(vii) If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta; k \in I$

(viii) $\arg(\overline{z}) = -\arg(z)$

(ix) $\arg(z - \overline{z}) = \pm\frac{\pi}{2}$

(x) $\arg(z) - \arg(-z) = \pm\pi$

Polar form (Trigonometric form) of a Complex Number

Let OP = r, then x = r cosθ, and y = r sinθ

z = x + iy = r cosθ + ir sinθ, = r(cos θ + i sinθ). This is known as Trigonometric (or Polar) form of a complex Number. Here we should take the principal value of θ.

For general values of the argument

z = r[cos (2nπ + θ) + i sin(2nπ + θ)] (where n is an integer)

Note : Sometimes cosθ + i sinq, in short is written as cis(θ).

Exponential form of a Complex Number (Euler's Form)

According to Euler's Theorem, $e^{i\theta}$ = cosθ + isinθ and therefore z = r(cosθ + isinθ) can be written as $z = re^{i\theta}$ which is called as exponential form of a complex number.

Replacing θ by –θ in $e^{i\theta}$, we obtain

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$\text{Hence } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\text{and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{y}{\sqrt{x^2 + y^2}}$$

Ex.1 Write the following complex numbers in polar and exponential form

(1) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ (2) 1 – i

Sol. (1) Here $r\cos\theta = -\frac{1}{2}, r\sin\theta = -\frac{\sqrt{3}}{2}$

Squaring and adding $r^2\cos^2\theta + r^2\sin^2\theta = \frac{1}{4}+\frac{3}{4} = 1$

$\therefore r = 1$ (-ve value is rejected)

Dividing we get $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$

Since $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ lies in third quadrant

Principal argument $= \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

$\therefore$ Polar form of $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ is $1\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)$

and Euler's form is $1.e^{-\frac{2\pi}{3}i}$

(2) Here $r\cos\theta = 1$ and $r\sin\theta = -1$

$\therefore r = \sqrt{2}, \tan\theta = -1 = \tan\left(-\frac{\pi}{4}\right)$

Since $(1,-1)$ lies in $IV^{è}$ quadrant, principal value of θ is $-\frac{\pi}{4}$

Polar form of $1-i$ is $\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}\times e^{-\frac{i\pi}{4}}$

REMEMBER

$1 = e^0, i = e^{\frac{i\pi}{2}}, -i = e^{-\frac{i\pi}{2}}, -1 = e^{i\pi}$

$\log i = \log e^{\frac{i\pi}{2}} = i\frac{\pi}{2}; \log(\log i) = \log\left(i\frac{\pi}{2}\right) = \log i + \log\frac{\pi}{2} = i\frac{\pi}{2} + \log\frac{\pi}{2}$