TRIGONOMETRIC FUNCTIONS

TRIGONOMETRIC FUNCTION OF SUM & DIFFERENCE OF TWO ANGLE

TRANSFORMATION FORMULAE

Product into sum and difference

1. $2\sin A\cos B = \sin(A+B) + \sin(A-B),\ A > B$
2. $2\cos A\sin B = \sin(A+B) - \sin(A-B),\ A > B$
3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Sum and Difference into products

1. $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
2. $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
3. $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
4. $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
5. $\tan C + \tan D = \frac{\sin(C+D)}{\cos C\cos D}$
6. $\tan C - \tan D = \frac{\sin(C-D)}{\cos C\cos D}$
7. $\cot C + \cot D = \frac{\sin(C+D)}{\sin C\sin D}$

8. $\cot C - \cot D = \dfrac{\sin(D-C)}{\sin C \sin D}$

Ex.1 Show that $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

Sol. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \dfrac{(\sin A + \sin 7A) + (\sin 3A + \sin 5A)}{(\cos A + \cos 7A) + (\cos 3A + \cos 5A)}$

$= \dfrac{2\sin 4A \cos 3A + 2\sin 4A \cos A}{2\cos 4A \cos 3A + 2\cos 4A \cos A}$

-tan4A

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES

T-Ratios of multiple angles : (An angle of the form nq, n $\in$ I)

1. $\sin 2A = 2 \sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

2. $\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

Thus,

$1 + \cos 2A = 2\cos^2 A$

$1 - \cos 2A = 2\sin^2 A$

3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}$

4. (i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}, \cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

5. $\cos A \cos 2A \cos^2 A$

$$\cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

T-Ratios of submultiple angle

(An angle of the from $\frac{\theta}{n}$, n ∈ l)

1. $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

2. $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

3. $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

4. $\cot\theta = \dfrac{\cot^2\frac{\theta}{2}-1}{2\cot\frac{\theta}{2}}$

5. $\cos^2\frac{\theta}{2} = \dfrac{1+\cos\theta}{2}$

6. $\sin^2\frac{\theta}{2} = \dfrac{1-\cos\theta}{2}$

7. $\tan^2\frac{\theta}{2} = \dfrac{1-\cos\theta}{1+\cos\theta}$

8. $\cot^2\frac{\theta}{2} = \dfrac{1+\cos\theta}{1-\cos\theta}$

9. $\dfrac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$

10. $\dfrac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$

T-Ratio of some special angles

(i) $\sin 15^\circ = \cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15^\circ = \sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15^\circ = \cot 75^\circ = 2-\sqrt{3}$

(iv) $\cot 15^\circ = \tan 75^\circ = 2+\sqrt{3}$

(v) $\sin 18^\circ = \cos 72^\circ = \dfrac{\sqrt{5}-1}{4}$

(vi) $\cos 18^\circ = \sin 72^\circ = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$

(vii) $\sin 36^\circ = \cos 54^\circ = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

(viii) $\cos 36^\circ = \sin 54^\circ = \dfrac{\sqrt{5}+1}{4}$

(ix) $\tan 22\dfrac{1}{2}^\circ = \sqrt{2}-1$

(x) $\cot 22\dfrac{1}{2}^\circ = \sqrt{2}+1$

Ex.2 Show that $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$

Sol. $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = \dfrac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ}$

$$= \frac{4\left(\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ\right)}{\sin 20^\circ} = \frac{4\left(\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ\right)}{\sin 20^\circ} = 4$$

Remember

(i) $\left|\sin\frac{A}{2}+\cos\frac{A}{2}\right| = \sqrt{1+\sin A}$

or $\sin\frac{A}{2}+\cos\frac{A}{2} = \pm\sqrt{1+\sin A}$

$$\text{i.e}\begin{cases} +, \text{ if } 2n\pi - \frac{\pi}{4} \le \frac{A}{2} \le 2n\pi + \frac{3\pi}{4} \\ -, \text{otherwise} \end{cases}$$

(ii) $\left|\sin\frac{A}{2}-\cos\frac{A}{2}\right| = \sqrt{1-\sin A}$

or $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right) = \pm\sqrt{1-\sin A}$

$$\text{i.e}\begin{cases} +, \text{ if } 2n\pi - \frac{\pi}{4} \le \frac{A}{2} \le 2n\pi + \frac{5\pi}{4} \\ -, \text{otherwise} \end{cases}$$

Greatest and least values of a cos θ + b sin θ

$S = a\cos q + b\sin q$

$= r\left(\frac{a}{r}\cos\theta + \frac{b}{r}\sin\theta\right): \quad r = \sqrt{a^2+b^2}$

$= r(\sin(\theta+\alpha)); \sin\alpha = \frac{a}{r}; \quad \cos\alpha = \frac{b}{r}$

Since $-1 \le \sin(\theta+\alpha) \le 1$, therefore, $-r \le S \le r$

Ex.3 Prove that $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3$ lies between -4 and 10.

Sol. $-5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$

$-5\cos\theta + 3\left(\frac{\cos\theta}{2} - \frac{\sqrt{3}\sin\theta}{2}\right) + 3$

$-\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$

Since $-7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$-4 \le S \le 10$

CONDITIONAL IDENTITIES

When, three angles A, B, C satisfy some given relation, several identities can be established connecting the trigonometric ratios of these angles

In a triangle ABC, A + B + C = p ;

sin (A + B) = sin (p – C) = sin C

and cos (A + B) = cos (p – C) = – cos C

Also, $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$; Hence,

$\sin\left(\frac{A+B}{2}\right) - \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) - \cos\frac{C}{2}$

$\cos\left(\frac{A+B}{2}\right) - \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) - \sin\frac{C}{2}$

Remember :

If A + B + C = p, then

(i) sin2A + sin2B + sin2C = 4sinA sinB sinC

(ii) cos2A + cos2B + cos2C = –1 – 4cosA cosB cosC

(iii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(iv) $\sin A + \sin B + \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(vii) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} - \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(viii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

Ex.4. If $A = \cos^2 q + \sin^4 q$, then for all values of q is

(1) $1 \le A \le 2$ (2) $\frac{13}{16} \le A \le 1$

(3) $\frac{3}{4} \le A \le \frac{13}{16}$ (4) $\frac{3}{4} \le A \le 1$

Sol. $A = \cos^2 q + \sin^2 q \sin^2 q$

$\Rightarrow A \le \cos^2 q + \sin^2 q.1$ $\left(\because \sin^2\theta \le 1\right)$

$\Rightarrow A \le 1$

Again, $A = (1 - \sin^2 q) + \sin^4 q$

$\Rightarrow A = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{4}\right) \Rightarrow A \ge \frac{3}{4}$

Hence $\frac{3}{4} \le A \le 1$

Ans. (4)

Ex.5 The value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$ is

(1) 0 (2) $\frac{1}{2}$

(3) $\frac{1}{3}$ (4) $-\frac{1}{8}$

Sol. $= -\dfrac{1}{2\sin\frac{\pi}{7}} \times \left(2\sin\dfrac{\pi}{7}\cos\dfrac{\pi}{7}\right)\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7}$

$$= -\frac{1}{2\sin\frac{\pi}{7}} \times \frac{1}{2}\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)\cos\frac{4\pi}{7}$$

$$= -\frac{1}{4\sin\frac{\pi}{7}} \times \frac{1}{2}\left(2\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}\right)$$

$$-\frac{1}{8} \times \frac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}} = \frac{\sin\left(\pi + \frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}}$$

$$= \frac{-\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{1}{8}$$

Ans. (4)

Ex.6 The period of the function, f (x) = 3 sin (2x + 1) in radians is

(1) 2p (2) p

(3) $\dfrac{\pi}{2}$ (4) –p

Sol. Period of sin x is 2p, The period of

f (x) = 3 sin (2x + 1) is $\dfrac{2\pi}{2} = \pi$

Alternatively we have

f (x) = 3 sin (2x + 1) = 3 sin (2p + 2x + 1)

= 3 sin {2(p + x) + 1} = f (p + x)

$\Rightarrow$ period of f (x) is p

Ans. (2)