TRIGONOMETRIC FUNCTIONS

TRIGONOMETRIC EQUATION

Trigonometric Equations

An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.

(i) Solution of Trigonometric Equation :

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. if $\sin\theta = \frac{1}{\sqrt{2}}$ $\Rightarrow$ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots\ldots\ldots\ldots$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as :

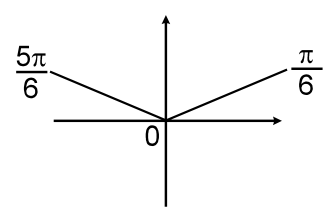
(a) Principal solution

(b) General solution.

(a) Principal solutions :

The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called Principal solutions. e.g. Find the Principal solutions of the equation $\sin x = \frac{1}{2}$.

$\because$ $\sin x = \frac{1}{2}$ $\because$ there exists two values



i.e. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$

$\therefore$ Principal solutions of the equation $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$

(b) General Solution :

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called General solution.

General solution of some standard trigonometric equations are given below.

General Solution of Some Standard Trigonometric Equations :

- If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.
- If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.
- If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.
- If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.
- If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.
- If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$.

  [**Note:** α is called the principal angle]

Some Important deductions :

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi,\quad n \in I$

(ii) $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2},\quad n \in I$

(iii) $\sin\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{2},\quad n \in I$

(iv) $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2},\quad n \in I$

(v) $\cos\theta = 1 \Rightarrow \theta = 2n\pi,\quad n \in I$

(vi) $\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi,\quad n \in I$

(vii) $\tan\theta = 0 \Rightarrow \theta = n\pi,\quad n \in I$

Ex.1 Solve $\sin\theta = \frac{1}{2}$.

Sol. $\because \quad \sin\theta = \frac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\frac{\pi}{6} \qquad \therefore \quad \theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

Ex.2 Solve $\sec 2\theta = -\frac{2}{\sqrt{3}}$

Sol. $\because \quad \sec 2\theta = -\frac{2}{\sqrt{3}}$

$\Rightarrow \quad \cos 2\theta = -\frac{\sqrt{3}}{2} \qquad \Rightarrow \quad \cos 2\theta = \cos\frac{5\pi}{6}$

$\Rightarrow \quad 2\theta = 2n\pi \pm \frac{5\pi}{6}, n \in I \qquad \Rightarrow \quad \theta = n\pi \pm \frac{5\pi}{12}, n \in I$

Ex.3 Solve $\tan\theta = 1$

Sol. $\because \quad \tan\theta = 1$ .... (i)

Let $1 = \tan\alpha$

$\Rightarrow \quad \tan\theta = \tan\frac{\pi}{4} \qquad \Rightarrow \quad \theta = n\pi + \frac{\pi}{4}, n \in I$

Ex.4 Solve $\sin^2\theta = \frac{3}{4}$

Sol. $\because \sin^2\theta = \frac{3}{4} \qquad \Rightarrow \quad \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3} \qquad \Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3}, n \in I$

Ex.5 Solve $4\cot 2\theta = \cot^2\theta - \tan^2\theta$

Sol. This equation is not defined for $\theta = \frac{n\pi}{2}$ where $n \in I$

$\frac{4}{\tan 2\theta} = \frac{1}{\tan^2\theta} - \tan^2\theta \qquad \Rightarrow \quad \frac{4(1-\tan^2\theta)}{2\tan\theta} = \frac{1-\tan^4\theta}{\tan^2\theta}$

or $\quad (1 - \tan^2\theta)\,[2\tan\theta - (1 + \tan^2\theta)]$

$\Rightarrow \quad (1 - \tan^2\theta)\,[\tan\theta - 1]^2 = 0$

$\tan^2\theta = 1$

$\tan^2\theta = \tan^2\frac{\pi}{4}$ $\theta = n\pi \pm \frac{\pi}{4}, n \in I$

(ii) Types of Trigonometric Equations :

Type -1 Trigonometric equations which can be solved by use of factorization.

Ex.6 Solve $(2\cos x - \sin x)(1 + \sin x) = \cos^2 x$.

Sol. $\because$ $(2\cos x - \sin x)(1 + \sin x) = \cos^2 x$

$\Rightarrow$ $(2\cos x - \sin x)(1 + \sin x) - \cos^2 x = 0$

$\Rightarrow$ $(2\cos x - \sin x)(1 + \sin x) - (1 - \sin x)(1 + \sin x) = 0$

$\Rightarrow$ $(1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow$ $1 + \sin x = 0$ or $2\cos x - 1 = 0$

$\Rightarrow$ $\sin x = -1$ or $\cos x = \frac{1}{2}$

$\Rightarrow$ $x = (4n - 1)\frac{\pi}{2}, n \in I$ or $\cos x = \cos\frac{\pi}{3}$

$\Rightarrow$ $x = 2n\pi \pm \frac{\pi}{3}$

$\therefore$ Solution of given equation is

$(4n - 1)\frac{\pi}{2}, n \in I$ or $2n\pi \pm \frac{\pi}{3}$

Type - 2 Trigonometric equations which can be solved by reducing them in quadratic equations.

Ex.7 Solve $2\cos^2\theta + 3\sin\theta = 0$

Sol. $2(1 - \sin^2\theta) + 3\sin\theta = 0$ $\Rightarrow$ $2\sin^2\theta - 3\sin\theta - 2 = 0$

$\Rightarrow$ $(\sin\theta - 2)(2\sin\theta + 1) = 0$ $\Rightarrow$ $\sin\theta \neq 2$

so $2\sin\theta + 1 = 0$ $\Rightarrow$ $\sin\theta = -\frac{1}{2}$

$\sin\theta = \sin\left(-\frac{\pi}{6}\right)$ $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$ or $\theta = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right)$

Type – 3 Trigonometric equations which can be solved by transforming a sum or difference of trigonometric ratios into their product.

Ex.8 Solve $\cos x + \cos 3x = 2\cos 2x$

Sol. $\cos x + \cos 3x = 2\cos 2x$

$\Rightarrow \quad 2\cos 2x \cos x - 2\cos 2x = 0 \qquad \Rightarrow \quad 2\cos 2x\,[\cos x - 1] = 0$

$\cos 2x = 0$ or $\cos x = 1$

$2x = (2n + 1)\,\frac{\pi}{2}$ or $x = 2m\pi,\ n, m \in Z$

$x = (2n + 1)\,\frac{\pi}{4}$ or $x = 2m\pi,\ n, m \in Z$

Type - 4 Trigonometric equations which can be solved by transforming a product of trigonometric ratios into their sum or difference.

Ex.9 Solve $\sin 9x.\cos 7x = \sin 10x.\cos 6x$

Sol. $\because \quad \sin 9x.\cos 7x = \sin 10x.\cos 6x$

$\Rightarrow \quad 2\sin 9x.\cos 7x = 2\sin 10x.\cos 6x$

$\Rightarrow \quad \sin 16x + \sin 2x = \sin 16x + \sin 4x$

$\Rightarrow \quad \sin 4x - \sin 2x = 0$

$\Rightarrow \quad 2\sin 2x.\cos 2x - \sin 2x = 0$

$\Rightarrow \quad \sin 2x\,(2\cos 2x - 1) = 0$

$\Rightarrow \quad \sin 2x = 0$ or $2\cos 2x - 1 = 0$

$\Rightarrow \quad 2x = n\pi,\ n \in I$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{n\pi}{2},\ n \in I$ or $2x = 2n\pi \pm \frac{\pi}{3},\ n \in I$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6},\ n \in I$

$\therefore$ Solution of given equation is

$\frac{n\pi}{2},\ n \in I$ or $n\pi \pm \frac{\pi}{6},\ n \in I$

Type – 5 Trigonometric Equations of the form $a \sin x + b \cos x = c$, where $a, b, c \in R$, can be solved by dividing both sides of the equation by $\sqrt{a^2+b^2}$.

Ex.10 Solve $\sqrt{3}\sin x + \cos x = \sqrt{3}$

Sol. $\because$ $\sqrt{3}\sin x + \cos x = \sqrt{3}$ .... (i)

Here $a = \sqrt{3}$, $b = 1$.

$\therefore$ divide both sides of equation (i) by 2, we get

$\text{Sin } x . \frac{\sqrt{3}}{2} + \cos x . \frac{1}{2} = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\sin x. \sin\frac{\pi}{3} + \cos x. \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\Rightarrow$ $\cos\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6}, n \in I$ $\Rightarrow$ $x = 2n\pi \pm \frac{\pi}{6} + \frac{\pi}{3}$, $n \in I$

$\therefore$ Solution of given equation is

$x = 2n\pi + \frac{\pi}{2}$ or $2n\pi + \frac{\pi}{6}$

Note : Trigonometric equation of the form $a \sin x + b \cos x = c$ can also be solved by changing sin x and cos x into their corresponding tangent of half the angle.

Ex.11 Solve $\cos x + \sqrt{3}\sin x = \sqrt{3}$ by changing sin x and cos x to there half angles

Sol. $\because$ $\cos x + \sqrt{3}\sin x = \sqrt{3}$ .... (i)

$\because$ $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\therefore$ equation (i) becomes

$\Rightarrow$ $\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + \sqrt{3}\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = \sqrt{3}$ .... (ii)

Let $\tan\frac{x}{2} = t$

$\therefore$ equation (ii) becomes $\left(\frac{1-t^2}{1+t^2}\right) + \sqrt{3}\left(\frac{2t}{1+t^2}\right) = \sqrt{3}$

$\Rightarrow$ $(\sqrt{3}+1)\,t^2 - 2\sqrt{3}\,t + \sqrt{3} - 1 = 0$

$\Rightarrow$ $(\sqrt{3}+1)\,t^2 - (\sqrt{3}+1)\,t + (1-\sqrt{3})t + \sqrt{3} - 1 = 0$

$\Rightarrow$ $[(\sqrt{3}+1)\,t + (1-\sqrt{3})]\,(t-1) = 0$

$\Rightarrow$ $\tan\frac{x}{2} = 1$ or $\tan\left(\frac{x}{2}\right) = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

so $x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi + \frac{\pi}{6}$

Type – 6 Trigonometric equations of the form $P(\sin x \pm \cos x, \sin x \cos x) = 0$, where $p(y, z)$ is a polynomial, can be solved by using the substitution $\sin x \pm \cos x = t$.

Ex.12 Solve $5(\sin x + \cos x) = 5 + 2\sin x.\cos x$

Sol. $\because$ $5(\sin x + \cos x) = 5 + 2\sin x.\cos x$ ....(i)

Let $\sin x + \cos x = t$

$\Rightarrow$ $\sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow$ $\sin x.\cos x = \frac{t^2-1}{2}$

Now put $\sin x + \cos x = t$ and $\sin x.\cos x = \frac{t^2-1}{2}$ in (i)

$\Rightarrow$ $t^2 - 5t + 4 = 0$ $\Rightarrow$ $t = 1$ or $t = 4$

$\Rightarrow$ $t \neq 4$ because $-\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}$

so $t = 1$

$\Rightarrow$ $\sin x + \cos x = 1$ .... (ii)

divide both sides of equation (ii) by $\sqrt{2}$, we get

$\Rightarrow$ $\sin x.\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$ $\Rightarrow$ $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

(i) if we take positive sign, we get $x = 2n\pi + \frac{\pi}{2}$, $n \in I$

(ii) if we take negative sign, we get $x = 2n\pi$, $n \in I$

Type – 7 Trigonometric equations which can be solved by the use of boundness of the trigonometric ratios sin x and cos x.

Ex.13 Solve $\cos^{50}x - \sin^{50}x = 1$

Sol. $\cos^{50}x - \sin^{50}x = 1$

$\Rightarrow$ $\cos^{50}x = \sin^{50}x + 1$

$\Rightarrow$ L.H.S. ≤ 1 & R.H.S. ≥ 1

so, both sides should be equal to 1

$\cos^{50}x = 1$ & $\sin^{50}x = 0$

or $\sin x = 0$ or $x = n\pi$ where $n \in I$

Important points :

- Many trigonometrical equations can be solved by different methods. The form of solution obtained in different methods may be different. From these different forms of solutions, the students should not think that the answer obtained by one method is wrong and those obtained by another method is correct. The solutions obtained by different methods may be shown to be equivalent by some supplementary transformations.

  To test the equivalence of two solutions obtained from two methods, the simplest way is to put values of n = ...–2, –1, 0, 1, 2, 3....... etc. and then to find the angles in $[0, 2\pi]$. If all the angles in both solutions are same, the solutions are equivalent.

- While manipulating the trigonometrical equation, avoid the danger of losing roots. Generally, some roots are lost by cancelling a common factor from the two sides of an equation. For Example, suppose we have the equation $\tan x = 2 \sin x$. Here by dividing both sides by sin x, we get $\cos x = \frac{1}{2}$. This is not equivalent to the original equation. Here the roots obtained by $\sin x = 0$, are lost. Thus in place of dividing an equation by a common factor, the students are advised to take this factor out as a common factor from all terms of the equation.

- While equating one of the factors to zero, take care of the other factor that it should not become infinite. For Example, if we have the equation sin x = 0, which can be written as cos x tan x = 0. Here we cannot put cos x = 0, since for cos x = 0, tan x = sin x/ cos x is infinite.
- Avoid squaring : When we square both sides of an equation, some extraneous roots appear. Hence it is necessary to check all the solutions found by substituting them in the given equation and omit the solutions not satisfying the given equation.

For Example : Consider the equation,

$\sin\theta + \cos\theta = 1$ .....(1)

Squaring we get

$1 + \sin 2\theta = 1$ or $\sin 2\theta = 0$ .....(2)

i.e. $2\theta = n\pi$ or $\theta = n\pi/2$,

This gives $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots\ldots$

Verification shows that π and $\frac{3\pi}{2}$ do not satisfy the equation as

$\sin\pi + \cos\pi = -1, \neq 1$ and $\sin\frac{3\pi}{2} + \cos\frac{3\pi}{2} = -1, \neq 1$.

The reason for this is simple.

The equation (2) is not equivalent to (1) and (2) contains two equations :

$\sin\theta + \cos\theta = 1$ and $\sin\theta + \cos\theta = -1$. Therefore we get extra solutions.

Thus if squaring is must, verify each of the solution.

- Some necessary restrictions :

If the equation involves tan x, sec x, take cos x $\neq$ 0. If cot x or cosec x appear, take Sin x $\neq$ 0.

If log appear in the equation, i.e. log [f (θ)] appear in the equation, use f (θ) > 0 and base of log > 0, $\neq$ 1.

Also note that $\sqrt{[f(\theta)]}$ is always positive, for Example $\sqrt{\sin^2\theta} = |\sin\theta|$, not $\pm\sin\theta$.

- Verification : Student are adviced to check whether all the roots obtained by them satisfy the equation and lie in the domain of the variable of the given equation.