RELATIONS AND FUNCTIONS

CARTESIAN PRODUCT, DOMAIN RANGE

RELATIONS

a R b means 'a is R-related to b' i.e. a is related to b under relation R. If $(a, b) \in R$; (a, b) is called ordered pair in the sense that a and b can't be interchanged as $a \in A$ and $b \in B$.

ORDERED PAIR :

It is a pair of objects written in a particular order. Two members are written in a particular order separated by a comma and enclosed in parentheses. Hence in ordered pair (a, b) a is called the first component or the first element or the first co-ordinate and b the second.

Ordered pairs (a, b) and (b, a) are different.

$(a, b) = (c, d)$ iff $a = c$ and $b = d$

i.e. $(1, 3) = (1, 3)$; $(1, 3) \neq (1, 2) \neq (2, 3) \neq (3, 1)$

CARTESIAN PRODUCT

Cartesian product of two sets A × B :

For any two non empty sets A and B

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

It is a set of all ordered pairs such that in each ordered pair first element belongs to set A and second element belongs to set B.

A × B is read as 'A cross B' or 'Product set of A and B'

$A \times B = \{(a, b) : a \in A \wedge b \in B\}$

Thus $(a, b) \in A \times B \Leftrightarrow a \in A$ and $b \in B$.

$B \times A = \{(b, a) : b \in B \wedge a \in A\}$

$A \times B \neq B \times A$ (not commutative)

$n(A \times B) = n(A)\, n(B)$ and $n(P(A \times B)) = 2^{n(A)\, n(B)}$

$A = \phi$ and $B = \phi \Leftrightarrow A \times B = \phi$

Cartesian product of n non empty sets $A_1, A_2, ...A_n$ is a set of all n tuples $(a_1, a_2, ...a_n)$ such that each $a_i \in A_i$, $i = 1, 2.........n$.

$$A_1 \times A_2 \times .... \times A_n = \prod_{i=1}^{n} A_i$$

$A \times A = A^2$: $R \times R = R^2$ is a set of all points lying in the plane

$R \times R \times R = R^3$ represents set of all points in 3-D space.

If at least one of A and B is infinite set then $A \times B$ is also infinite set, provided that other is non-empty set.

Ex.1 Let $A = \{a, b\}$, $B = \{c, d\}$, $C = \{e, f\}$

Sol. then $n(A \times B \times C) = n(A).\ n(B).\ n(C) = 8$

$A \times B \times C = \{(a, c, e), (a, c, f), (a, d, e), (a, d, f), (b, c, e), (b, c, f), (b, d, e), (b, d, f)\}$

Note: Each element of $A \times B$ shall be an ordered pair or 2–tuple

Each element of $A \times B \times C$ shall be an ordered triplet or 3–tuple

Each element of $A_1 \times A_2 \times .....A_n$ shall be n–tuple

Ex.2 $A_1 \times A_2 \times A_3 \times A_4 = \{(1, 1, 1, 1), (2, 4, 8, 16), (3, 9, 27, 81), ...\}$. Find A_1, A_2, A_3 and A_4 .

Sol. Each ordered pair $\{x_1, x_2, x_3, x_4\}$ is of the form $\{x, x^2, x^3, x^4\}$

Hence $x_1 \in A_1 \Rightarrow A_1 = \{x : x \in N\} = \{1, 2, 3, 4, ...\}$

$x_2 \in A_2 \Rightarrow A_2 = \{x^2 : x \in N\} = \{1^2, 2^2, 3^2, 4^2, ...\}$

$x_3 \in A_3 \Rightarrow A_3 = \{x^3 : x \in N\} = \{1^3, 2^3, 3^3, 4^3, ...\}$

$x_4 \in A_4 \Rightarrow A_4 = \{x^4 : x \in N\} = \{1^4, 2^4, 3^4, 4^4, ...\}$

Key Results on Cartesian Product

If A, B, C, D are four sets, then.

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
5. If $A \subseteq B$, then $(A \times C) \subseteq (B \times C)$
6. If $A \subseteq B$, then $(A \times B) \cap (B \times A) = A^2$
7. If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
8. If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$
9. $A \times B = B \times A \Leftrightarrow A = B$
10. $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
11. $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

If A and B are two non-empty sets having n elements in common then $(A \times B)$ and $(B \times A)$ have n^2 elements in common.

Ex.3 If $n(A) = 7$, $n(B) = 8$ and $n(A \cap B) = 4$, then match the following columns.

(i)	$n(A \cup B)$	(a)	56
(ii)	$n(A \times B)$	(b)	16
(iii)	$n((B \times A) \times A)$	(c)	392
(iv)	$n((A \times B) \cap (B \times A))$	(d)	96
(v)	$n((A \times B) \cup (B \times A))$	(e)	11

Sol. (i)–(e); (ii)–(a); (iii)–(c); (iv)–(b); (v)–(d)

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 7 + 8 - 4 = 11$

(ii) $n(A \times B) = n(A)\, n(B) = 7 \times 8 = 56 = n(B \times A)$

(iii) $n((B \times A) \times A) = n(B \times A).\, n(A) = 56 \times 7 = 392$

(iv) $n((A \times B) \cap (B \times A)) = (n((A \cap B))^2 = 4^2 = 16$

(v) $n((A \times B) \cup (B \times A)) = n(A \times B) + n(B \times A) - n(A \times B) \cap (B \times A)$

$= 56 + 56 - 16 = 96$

Ex.4 If $A = \{2, 4\}$ and $B = \{3, 4, 5\}$, then $(A \cap B) \times (A \cup B)$ is

(1) $\{(2, 2), (3, 4), (4, 2), (5, 4)\}$

(2) $\{(2, 3), (4, 3), (4, 5)\}$

(3) $\{(2, 4), (3, 4), (4, 4), (4, 5)\}$

(4) $\{(4, 2), (4, 3), (4, 4), (4, 5)\}$

Sol. Answer (4)

$A \cap B = \{4\}$ and $A \cup B = \{2, 3, 4, 5\}$

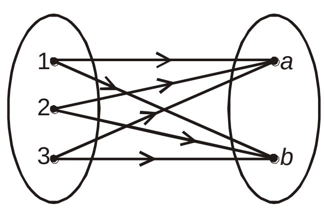
$(A \cap B) \times (A \cup B) = \{(4, 2), (4, 3), (4, 4), (4, 5)\}$

Pictorial Representation of Cartesian Product of Two Sets :

Arrow diagram :

Let $A = \{1, 2, 3\}$ $B = (a, b)$

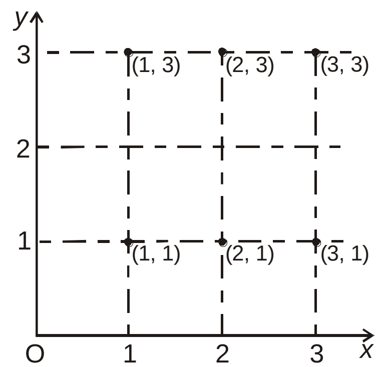
$A \times B$



Lattice-Diagram :

Axis OX represents elements of A and perpendicular axis OY represents set B. Each dot represents an ordered pair of $A \times B$.

Let $A = (1, 2, 3)$ $B = (1, 3)$



RELATIONS

For any two non-empty sets A and B, every subset of A × B defines a relation from A to B and every relation from A to B is a subset of A × B.

$a\ R\ b \subseteq A \times B \ \ \forall R$

If $(a, b) \in R$, then a R b is read as 'a is related to b'

If $(a, b) \notin R$, then a $\not{R}$ b is read as 'a is not related to b'

Domain and Range of Relation

Domain of R = Dom(R) = Set of first components of all the ordered pairs belonging to R.

Range of R = Set of second components of all the ordered pairs belonging to R.

Co-domain of R = B where R is a relation from A to B

Range of R $\subseteq$ Co-domain of R

$\text{Dom}(R) = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$

Range of $R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$

If R = A × B, then Dom(R) = A and Range of R = B

Dom (f) = ϕ ; Range of f = ϕ

Ex.5 Let A = {1, 3, 4, 5, 7} and B = {1, 4, 6, 7} and R be the relation 'is one less than' from A to B, then list the domain, range and co-domain sets of R.

Sol. R = {(3, 4), (5, 6)}

So, Dom(R) = {3, 5}

Range of R = {4, 6}

Co-domain of R = B = {1, 4, 6, 7}

Clearly Range of R $\subseteq$ co-domain of R.

Representation of a Relation

Let A = {–2, –1, 4} B = {1, 4, 9}

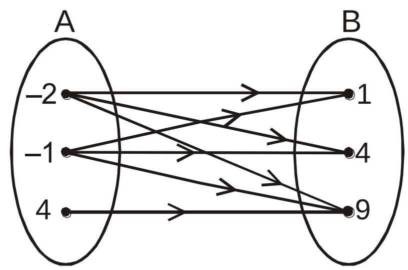
A relation from A to B i.e. a R b is defined as a is less than b.

This can be represented in the following ways.

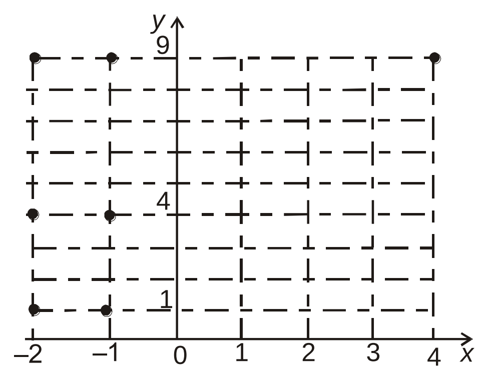
1. **Roster form:** R = {(–2, 1), (–2, 4), (–2, 9), (–1, 1), (–1, 4), (–1, 9), (4, 9)}

2. **Set builder notation:** R = {(a, b): a $\in$ A and b $\in$ B, a is less than b}

3. **Arrow - diagram:**



4. **Lattice-diagram :**



5. **Tabular form:**

R	1	4	9
-2	1	1	1
-1	1	1	1
4	0	0	1

Note : If (a, b) $\in$ R, then we write '1' in the row containing a and column containing b and if (a, b) $\notin$ R, we write '0' in the respective row and column.

Ex.6 Let A = {1, 2, 3, 4}, B = {1, 2, 3,.......10}

R_1 = {(1, 4), (2, 5), (3, 6), (4, 7)}

R_2 = {(2, 5), (3, 6), (4, 7), (5, 8)}

$R_3 = \{(1, 1), (2, 4), (3, 9)\}$

Among R_1, R_2, R_3, choose those, that represent a relation from A to B, and represent the relations in set-builder form.

Sol. $R_1 \subseteq A \times B$; $R_2 \not\subseteq A \times B$, $R_3 \subseteq A \times B$

Hence R_2 is not a relation as $(5, 8) \notin A \times B$

$R_1 = \{(a, b) : a \in A, b \in B \text{ and } a + 3 = b\}$

$R_3 = \{(a, b) : a \in A, b \in B \text{ and } a^2 = b\}$

Any subset of $A \times A$ is a relation on A. If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$

Total number of subsets of $(A \times B) = 2^{pq}$

Hence 2^{pq} different relations are possible from A to B.