MATHEMATICAL REASONING

STATEMENTS

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language. "A sentence is called a mathematically acceptable statement or proposition if it is either true or false but not both".

A statement is assumed to be either true or false. A true statement is known as a **valid statement** and a false statement is known as an **invalid statement**.

A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.

If a statement is true then its truth value is T and if it is false then its truth value is F

Ex. **(i)** "New Delhi is the capital of India", a true statement

(ii) "3 + 2 = 6", a false statement

(iii) "Where are you going ?" not a statement because it cannot be defined as true or false

❖ A statement cannot be both true and false at a time

Ex. Which of the following sentences are statements :

(i) Three plus two equals five.

(ii) The sum of two negative number is negative

(iii) Every square is a rectangle.

Sol. Each of these sentences is a true sentence therefore they all are statements.

Simple Statement

Any statement whose truth value does not depend on other statement are called simple statement

Ex. **(i)** " $\sqrt{2}$ is an irrational number" **(ii)** "The set of real number is an infinite set"

Open Statement

A sentence which contains one or more variable such that when certain values are given to the variable it becomes a statement, is called an open statement.

e.g. P : 'He is a great man' is an open statement because in this statement, we can be replaced by any person.

Compound Statements

If a statement is combination of two or more statements, then it is said to be a compound statement.

Each statement which form a compound statement is known as its sub-statement or component statement.

For Ex.

(i) "If x is divisible by 2 then x is even number"

(ii) "ΔABC is equilateral if and only if its three sides are equal"

Ex. Which of the following sentences are statements :

(i) Give me a glass of water. **(ii)** Is every set finite ?

(iii) How beautiful ? **(iv)** Tomorrow is Monday.

(v) May God bless you !

Sol. None of these sentences is a statement

(i) Imperative (expresses a request or command), exclamatory sentences (expresses some strong feeling), Interrogative sentences (asks some question) are not considered as a statement in mathematical language.
(ii) Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements.
(iii) Scientifically established facts are considered true.
(iv) Optative (blessing & wishes) sentences are not a statement.

TRUTH TABLE

A table which shows the relationship between the truth value of compound statement S(p, q, r .....) and the truth values of its sub statements p, q, r, ... is said to be truth table of compound statement S.

Truth table is that which gives truth values of statements. It has a number of rows and columns.

Note that for n statements, there are 2^n rows.

(i) Truth table for single statement p:
Number of rows $= 2^1 = 2$

p
T
F

(ii) Truth table for two statements p and q :

p	q
T	T
T	F
F	T
F	F

Number of rows $= 2^2 = 4$

(iii) Truth table for three statements p, q and r.
Number of rows $= 2^3 = 8$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

NEGATION OF A STATEMENT

p	~p
T	F
F	T

Truth table

The denial of a statement p is called its negation and is written as ~ p and read as 'not p'. Negation of any statement p is formed by writing "It is not the case that ...."

or "It is false that . . . ."

or inserting the word "not" in p.

Ex. Write negation of following statements :

(i) "All cats scratch" (ii) "$\sqrt{5}$ is a rational number".

Sol. (i) Some cats do not scratch

OR

There exists a cat which does not scratch

OR

At least one cat does not scratch

(ii) $\sqrt{5}$ is an irrational number

USE OF VENN–DIAGRAMS FOR FINDING TRUTH VALUES OF STATEMENTS

We are familiar with Venn diagrams. These diagrams are used very frequently in the problems of set theory. Venn diagrams can also be used for deciding the truthfulness of statements.

Ex. Represent the truth of each of the following statements by means of a Venn–diagram :

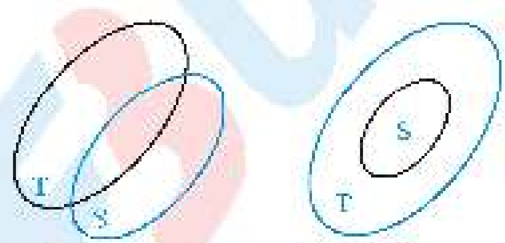
(i) Some teachers are scholars.

(ii) Some quadratic equations have two real roots.

(iii) All human beings are mortal and x is not a human being.

Sol. (i) Let T = Set of all teachers and S = set of all scholars.

Since the given statement 'some teachers are scholars' is true, we have $T \cap S \neq \phi$ and $T \cap S \subseteq S$.



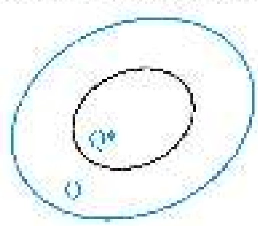
$T \cap S \subset S$ $\quad$ $T \cap S = S$

$\therefore$ Either $T \cap S \subset S$ or $T \cap S = S$.

The truth of the given statement is shown in the adjoining Venn diagrams :

(ii) Let Q = set of all quadratic equations and Q* = set of all quadratic equations having real roots.

Since the given statement 'some quadratic equations have two real roots is true, we have $Q^* \subset Q$.



The truth of the given statement is shown in the adjoining Venn diagram.

(iii) Let H = set of all human being

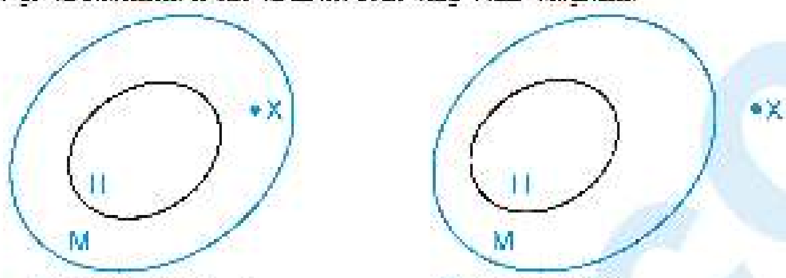
and M = set of all mortals.

Since the given statement 'all human beings are mortal and x is not a human being' is true,

we have

(i) $H \subset M$, $x \in M - H$ or (ii) $H \subset M$, $x \notin M$

The truth of the given statement is shown in the following Venn diagrams.



(i) $H \subset M$, $x \in M - H$ (ii) $H \subset M$, $x \notin M$.

LOGICAL CONNECTIVES

In the compound statement, two or more statements are connected by words like 'and', 'or', 'if ..... then', 'only if', 'if and only if', 'there exists', 'for all' etc. These are called connectives. When we use these compound statements, it is necessary to understand the role of these words.

The Word "AND" (CONJUNCTION)

Any two statements can be connected by the word "and" to form a compound statement. The compound statement with word "and" is true if all its component statements are true. The compound statement with word "and" is false if any or all of its component statements are false. The compound statement "p and q" is denoted by "$p \wedge q$".

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table

The Word "OR" (DISJUNCTION)

Any two statements can be connected by the word "OR" to form a compound statement. The compound statement with word "or" is true if any or all of its component statements are true. The compound statement with word "or" is false if all its component statement are false. The compound statement "p or q" is denoted by "$p \vee q$".

p	q	$p \wedge q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth table

Types of "OR"

(i) **Exclusive OR :** If in statement $p \vee q$ i.e. p or q, happening of any one of p, q excludes the happening of the other then it is exclusive or. Here both p and q cannot occur together. For example in statement "I will go to delhi either by bus or by train", the use of 'or' is exclusive.

(ii) **Inclusive OR :** If in statement p or q, both p and q can also occur together then it is inclusive or. The statement 'In senior secondary exam. you can take optional subject as physical education or computers' is an example of use of inclusive OR.

Implication

There are three types of implications which are "if ..... then", "only if" and "if and only if".

Conditional Connective "IF ..... THEN"

If p and q are any two statements then the compound statement in the form "If p then q" is called a conditional statement. The statement "If p then q" is denoted by $p \Rightarrow q$ or $p \rightarrow q$ (to be read as p implies q). In the implication $p \Rightarrow q$, p is called the antecedent (or the hypothesis) and q the consequent (or the conclusion).

If p then q reveals the following facts :

(i) p is a sufficient condition for q

(ii) q is a necessary condition for p

(iii) "If p then q" has same meaning as that of "p only if q"

(iv) $p \rightarrow q$ has same meaning as that of $\sim q \rightarrow \sim p$

p	q	p → q	q → p
T	T	T	T
T	F	T	T
F	T	T	F
F	F	T	T

Truth table

Ex. (i) If $x = 4$, then $x^2 = 16$

(ii) If ABCD is a parallelogram, then AB = CD

(iii) If Mumbai is in England, then $2 + 2 = 5$

(iv) If Shikha works hard, then it will rain today.

Contrapositive, Contradiction and Converse of a Conditional Statement

If p and q are two statements then

Let $p \Rightarrow q$ Then

(i) (Contrapositive of $p \Rightarrow q$) is $(\sim q \Rightarrow \sim p)$

(ii) (Contradiction of $p \Rightarrow q$) is $(q \Rightarrow \sim p)$

(iii) (Converse of $p \Rightarrow q$) is $(q \Rightarrow p)$

❖ A statement and its contrapositive convey the same meaning.

Biconditional Connective "IF AND ONLY IF"

If p and q are any two statements then the compound statement in the form of "p if and only if q" is called a biconditional statement and is written in symbolic form as $p \leftrightarrow q$ or $p <> q$.

Statement $p <> q$ reveals the following facts :

(i) p if and only if q

(ii) q if and only if p

(iii) p is necessary and sufficient condition for q

(iv) q is necessary and sufficient condition for p

p	q	p <> q	q <> p
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Truth table

Ex. Find the truth value of the statement "2 divides 4 and 3 + 7 = 8"

Sol. 2 divides 4 is true and 3 + 7 = 8 is false. so given statement is false.

Ex. Write component statements of the statement "All living things have two legs and two eyes".

Sol. Component statements are :

All living things have two legs

All living things have two eyes

Ex. Which of the following is correct for the statements p and q ?

(1) $p \wedge q$ is true when at least one from p and q is true

(2) $p \to q$ is true when p is true and q is false

(3) $p \leftrightarrow q$ is true only when both p and q are true

(4) $\sim(p \vee q)$ is true only when both p and q are false

Sol. We know that $p \wedge q$ is true only when both p and q are true so option (1) is not correct

we know that $p \to q$ is false only when p is true and q is false so option (2) is not correct

we know that $p \leftrightarrow q$ is true only when either p and q both are true or both are false

so option (3) is not correct

we know that $\sim(p \vee q)$ is true only when $(p \vee q)$ is false

i.e. p and q both are false

So option (4) is correct

Ex. Write the contrapositive of the following statement : "If Mohan is poet, then he is poor".

Sol. Consider the following statements :

p : Mohan is a poet

q : Mohan is poor

Clearly, the given statement in symbolic form is $p \to q$. Therefore, its contrapositive is given by $\sim q \to \sim p$.

Now, $\sim p$: Mohan is not a poet.

$\sim q$: Mohan is not poor.

$\therefore$ $\sim q \to \sim p$: If Mohan is not poor, then he is not a poet.

Hence the contrapositive of the given statement is "If Mohan is not poor, then he is not a poet".

Ex. Write the converse and the contrapositive of the statement "If x is a prime number, then x is odd".

Sol. Given statement is : "If x is a prime number then x is odd".

Let p : x is a prime number and q : x is odd

$\therefore$ Given statement is $p \to q$

The converse of $p \to q$ is $q \to p$ i.e. "If x is odd then x is a prime number"

The contrapositive of $p \to q$ is $\sim q \to \sim p$ i.e. "If x is not odd then x is not a prime number".

Ex. Write the contradiction of "If it rains, then I stay" at home.

Sol. If I stay at home then it does not rain.

Ex. Let p and q stand for the statement 'Bhopal is in M.P.' and '3 + 4 = 7' respectively. Describe the conditional statement $\sim p \to \sim q$.

Sol. $\sim p \to \sim q$: If Bhopal is not in M.P. then $3 + 4 \neq 7$.

Ex. Find the truth values of $(p \leftrightarrow \sim q) \leftrightarrow (q \to p)$.

Sol.

p	q	~ q	p ↔ ~ q	q → p	(p ↔ ~ q) ↔ (q ↔ p)
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	T

LOGICALLY EQUIVALENT STATEMENTS

Two compound statements $S_1(p, q, r...)$ and $S_2(p, q, r....)$ are said to be logically equivalent or simply equivalent if they have same truth values for all logically possibilities.

Two statements S_1 and S_2 are equivalent if they have identical truth table i.e. the entries in the last column of their truth table are same. If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

For Ex. The truth table for $(p \to q)$ and $(\sim p \vee q)$ given as below

p	q	$(\sim p)$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that last two columns of the above truth table are identical hence compound statements

$(p \to q)$ and $(\sim p \vee q)$ are equivalent

i.e. $\quad p \to q \equiv \sim p \vee q$

Ex. Equivalent statement of the statement "if $8 > 10$ then $2^3 = 5$".

Sol. We know that $p \to q \equiv \sim p \vee q$

$\therefore$ equivalent statement will $8 \leq 10$ or $2^3 = 5$

or $\quad 8 \leq 10$ or $2^3 = 5$

TAUTOLOGY AND FALLACY / CONTRADICTION

(i) **Tautology :** A statement is said to be a tautology if it is true for all logical possibilities

i.e. its truth value always T. it is denoted by t.

For Ex. the statement $p \vee \sim (p \wedge q)$ is a tautology

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Clearly. The truth value of $p \vee \sim (p \wedge q)$ is T for all values of p and q. so $p \vee \sim (p \wedge q)$ is a tautology

(ii) **Contradiction :** A statement is a contradiction if it is false for all logical possibilities.

i.e. its truth value always F. It is denoted by c.

For Ex. The statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p \wedge \sim q)$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Clearly. then truth value of $(p \vee q) \wedge (\sim p \wedge \sim q)$ is F for all value of p and q. So $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

- The negation of a tautology is a contradiction and negation of a contradiction is a tautology

ALGEBRA OF STATEMENTS

If p, q, r are any three statements then the some law of algebra of statements are as follow

(i) Idempotent Laws

(A) $p \wedge p = p$ **(B)** $p \vee p = p$

i.e. $p \wedge p = p = p \vee p$

p	$(p \wedge p)$	$(p \vee p)$
T	T	T
F	F	F

(ii) Commutative laws

(A) $p \wedge q = q \wedge p$ **(B)** $p \vee q = q \vee p$

p	q	$(p \wedge q)$	$(q \wedge p)$	$(p \vee q)$	$(q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

(iii) Associative laws

(A) $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

(B) $(p \vee q) \vee r = p \vee (q \vee r)$

p	q	r	$(p \wedge q)$	$(q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Similarly we can proved result **(B)**

(iv) Distributive laws : **(A)** $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ **(C)** $\wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$

(B) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ **(D)** $p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$

p	q	r	$(q \vee r)$	$(p \wedge q)$	$(p \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Similarly we can prove result **(B)**, **(C)**, **(D)**

(v) **De Morgan Laws :** (A) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(B) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	~p	~q	(p ∧ q)	~(p ∧ q)	(~p ∨ ~q)
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Similarly we can proved result (B)

(vi) **Involution laws (or Double negation laws) :** $\sim(\sim p) \equiv p$

p	~p	~(~p)
T	F	T
F	T	F

(vii) **Identity Laws :** If p is a statement and t and c are tautology and contradiction respectively then

(A) $p \wedge t \equiv p$ (B) $p \vee t \equiv t$ (C) $p \wedge c \equiv c$ (D) $p \vee c \equiv p$

p	t	c	(p ∧ t)	(p ∨ t)	(p ∧ c)	(p ∨ c)
T	T	F	T	T	F	T
F	T	F	F	T	F	F

(viii) **Complement Laws**

(A) $p \wedge (\sim p) \equiv c$ (B) $p \vee (\sim p) \equiv t$ (C) $(\sim t) \equiv c$ (D) $(\sim c) \equiv t$

p	~p	(p ∧ ~p)	(p ∨ ~p)
T	F	F	T
F	T	F	T

(ix) **Contrapositive laws :** $p \rightarrow q \equiv \sim q \rightarrow \sim p$

p	q	~p	~q	p → q	~q → ~p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(x) **Biconditional Statement**

p	q	p ↔ q	q ↔ p	p → q	q → p	p ↔ q ≡ (p → q) ∧ (q → p)
T	T	T	T	T	T	T
T	F	F	F	F	T	F
F	T	F	F	T	F	F
F	F	T	T	T	T	T

Ex. $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to.

Sol. $\sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ (By Demorgan Law)

$\equiv \sim p \wedge (\sim q \vee q)$ (By distributive laws)

$\equiv \sim p \wedge t$ (By complement laws)

$\equiv \sim p$ (By Identity Laws)

NEGATION OF COMPOUND STATEMENTS

If p and q are two statements then

(i) **Negation of Conjunction :** $\sim(p \wedge q) = \sim p \vee \sim q$

p	q	~ p	~ q	(p ∧ q)	~ (p ∧ q)	(~ p ∨ ~ q)
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(ii) **Negation of Disjunction :** $\sim(p \vee q) = \sim p \wedge \sim q$

p	q	~ p	~ q	(p ∨ q)	~ (p ∨ q)	(~ p ∧ ~ q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(iii) **Negation of Conditional :** $\sim(p \rightarrow q) = p \wedge \sim q$

p	q	~ q	(p → q)	~ (p → q)	(p ∧ ~ q)
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(iv) **Negation of Biconditional :** $\sim(p \leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$ or $p \leftrightarrow \sim q$

p	q	~ p	~ q	(p → q)	~ (p → q)	(p ∧ ~ q)	(p ↔ q)	~ (p ↔ q)	p ↔ ~ q	q ∧ ~ p	(q ∧ ~ q) ∨ (q ∧ ~ p)
T	T	F	F	T	F	F	T	F	F	F	F
T	F	F	T	F	T	T	F	T	T	F	F
F	T	T	F	T	F	F	F	T	T	T	T
F	F	T	T	T	F	F	T	F	F	F	F

We know that $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

$\therefore \quad \sim(p \leftrightarrow q) = \sim[(p \rightarrow q) \wedge (q \rightarrow p)]$

$= \sim(p \rightarrow q) \vee \sim(q \rightarrow p) = (p \wedge \sim q) \vee (q \wedge \sim p)$

(i) $\sim(p \wedge q) = (\sim p) \vee (\sim q)$

(ii) $\sim(p \vee q) = (\sim p) \wedge (\sim q)$

(iii) $\sim(p \Rightarrow q) = \sim(\sim p \vee q) = p \wedge (\sim q)$

(iv) $\sim(p \Leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$ or $p \Leftrightarrow \sim q$

Ex. Write Negation of the statement $p \to (q \wedge r)$.

Sol. $\sim(p \to (q \wedge r)) \equiv p \wedge \sim(q \wedge r)$ (✈ $\sim(p \to q) \equiv p \wedge \sim q$)

$\equiv p \wedge (\sim q \vee \sim r)$

Ex. Write the negation of the following compound statements :

(i) All the students completed their homework and the teacher is present.

(ii) Square of an integer is positive or negative.

(iii) If my car is not in workshop, then I can go college.

(iv) ABC is an equilateral triangle if and only if it is equiangular

Sol. (i) The component statements of the given statement are :

p : all the students completed their homework.

q : The teacher is present.

The given statement is p and q. so its negation is ~p or ~q = Some of the students did not complete their home work or the teacher is not present.

(ii) The component statement of the given statements are :

p : Square of an integer is positive.

q : Square of an integer is negative.

The given statement is p or q. so its negation is ~p and ~q = Their exists an integer whose square is neither positive nor negative.

(iii) Consider the following statements :

p : My car is not in workshop

q : I can go to college

The given statement in symbolic form is $p \to q$

Now, $\sim(p \to q) \cong p \wedge (\sim q)$

$\Rightarrow$ $\sim(p \to q)$: My car is not in workshop and I cannot go to college.

Hence the negation of the given statements is “My car is not in workshop and i can not go to college”.

(iv) Consider the following statements :

p : ABC is an equilateral triangle.

q : It is equiangular

Clearly, the given statement is symbolic form is $p \leftrightarrow q$.

Now, $\sim(p \leftrightarrow q) \cong (p \wedge \sim q) \vee (\sim p \wedge q)$

$\Rightarrow$ $\sim(p \to q)$: Either ABC is an equilateral triangle and it is not equiangular or ABC is not an equilateral triangle and it is equiangular.

Ex. The negation of the statement "If a quadrilateral is a square then it is a rhombus"

Sol. Let p and q be the statements as given below

p : a quadrilateral is a square

q : a quadrilateral is a rhombus

the given statement is $p \to q$

✈ $\sim(p \to q) \equiv p \wedge \sim q$

Therefore the negation of the given statement is a quadrilateral is a square and it is not a rhombus

DUALITY

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$

If a compound statement contains the special variable t (tautology) and c (contradiction) then obtain its dual we replaced t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

(i) Duality of Connectives : The connectives $\wedge$ and $\vee$ are called duals of each other.

(ii) Duality of Compound Statements : Two compound statements are called duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ $\vee$, $\vee$ by $\wedge$, tautology t by fallacy f and fallacy f by tautology t.

Ex. **(A)** The compound statements $(p \vee q) \wedge r$ and $(p \wedge q) \vee r$ are duals of each other.

(B) The compound statements $(p \wedge q) \vee (r \vee t)$ and $(p \vee q) \wedge (r \wedge f)$ are duals of each other.

(iii) Duality of Logical Equivalences : Two logical equivalences are called duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Ex. **(A)** The logical equivalences $\sim(p \wedge q) \equiv \sim p \text{ v} \sim q$ and $\sim(p \vee q) \sim p \wedge \sim q$ are duals of each other.

(B) The logical equivalences $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ are duals of each other.

(i) The connectives $\wedge$ and $\vee$ are also called dual of each other.

(ii) If $S^*(p, q)$ is the dual of the compound statement $S(p, q)$ then

(A) $S^*(\sim p, \sim q) \equiv \sim S(p, q)$ **(B)** $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

(iii) Let S(p, q, r,) be a compound statement in terms of finitely many statements p, q, r, If S* (p, q, r,) be the dual compound statement of S(p, q, r,………), then $\sim S^*(p, q, r, \ldots\ldots) \equiv S(\sim p, \sim q, r, \ldots\ldots..)$.

Ex. The duals of the following statements

(i) $(p \wedge q) \vee (r \vee s)$ **(ii)** $(p \vee t) \wedge (p \vee c)$ **(iii)** $\sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim s)]$

Sol. **(i)** $(p \vee q) \wedge (r \wedge s)$ **(ii)** $(p \wedge c) \vee (p \wedge t)$ **(iii)** $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim s)]$

VALIDITY OF A STATEMENT

There are four methods to prove validity of a statement.

(A) Direct Method

(i) To prove that “p and q” is true, show that both p and q are true

(ii) To prove “p or q”, show that any one of p or q is true.

(iii) To prove $p \rightarrow q$, assume that p is true and show that q must be true.

(iv) To prove $p \leftrightarrow q$, show that if p is true then q is true. Also show that if q is true, then p is true.

(B) Contrapositive Method

To prove $p \rightarrow q$, assume that q is false and prove that p must be false.

(C) Contradiction Method

To prove that a statement p is true, we assume that p is not true, then we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.

(D) Counter Example Method

To show that a statement is false, we give an example where the statement is not valid. Note that this method is used to disprove the statement. Giving examples in favour of a statement cannot prove that the given statement is valid.

CONVERSE, INVERSE AND CONTRAPOSITIVE OF THE CONDITIONAL STATEMENT ($p \rightarrow q$)

(i) Converse : The converse of the conditional statement $p \rightarrow q$ is defined as $q \rightarrow p$

(ii) Inverse : The inverse of the conditional statement $p \rightarrow q$ is defined as $\sim p \rightarrow \sim q$

(iii) Contrapositive : The contrapositive of conditional statement $p \rightarrow q$ is defined as $\sim q \rightarrow \sim p$

Ex. If $x = 5$ and $y = -2$ then $x - 2y = 9$. Write the contrapositive of this statement.

Sol. Let p, q, r be the three statements such that

$$p : x = 5, \quad q : y = -2 \text{ and } r : x - 2y = 9$$

Here given statement is $(p \wedge q) \rightarrow r$ and its contrapositive is $\sim r \rightarrow \sim(p \wedge q)$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$

VALIDITY OF AN ARGUMENT

An argument is an assertion that a given set of statements $s_1, s_2, ... s_n$ implies other statement 's'. In other words, an argument is an assertion that the statement 's' follows from statements $s_1, s_2, ... s_n$ which are called hypotheses. The statement 's' is called the conclusion.

We denote the argument containing hypotheses $s_1, s_2, ... s_n$ and conclusion 's' by

$s_1, s_2, ... s_n$; s or

$s_1, s_2 ..... s_n$ /- s or

$(s_1 \wedge s_2 \wedge ..... \wedge s_n) \rightarrow s$ or

s_1
s_2
s_3
...
...
s_n
_____
so s

The symbol "/-" is read as turnstile.

An argument is said to be a valid argument if the conclusion 's' is true whenever all the hypotheses $s_1, s_2, ... s_n$ are true or equivalently argument is valid when it is a tautology, otherwise the argument is called an invalid argument.

Method of Testing the Validity of Argument

Step I - Construct the truth table for conditional statement $S_1 \wedge S_2 \wedge S_3 \wedge .... \wedge S_n \rightarrow S$.

Step II - Check the last column of truth table. If the last column contains T only, then the given argument is valid otherwise it is an invalid argument.

Ex. Show that the following argument is not valid : "If it rains, crops will be good. It did not rain. Therefore the crops were not good".

Sol. p : it rains

q : crops will be good

$S_1 : p \rightarrow q$, $S_2 : \sim p$ $S : \sim q$

p	q	S_1	S_2	S
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Not valid

• Etoos Tips & Formulas •

1. Statement

A sentence which is either true or false but cannot be both are called a statement. A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.
If a statement is true then its truth value is T and if it is false then its truth value is F

Simple Statement

Any statement whose truth value does not depend on other statement are called simple statement

Compound Statements

If a statement is combination of two or more statements, then it is said to be a compound statement.
Each statement which form a compound statement is known as its sub statement or component statement.

Open Statement

A sentence which contains one or more variable such that when certain values are given to the variable it becomes a statement, is called an open statement.

2. Truth Table

A table which shows the relationship between the truth value of compound statement S(p, q, r ....) and the truth values of its sub statements p, q, r, ... is said to be truth table of compound statement S.
Truth table is that which gives truth values of statements. It has a number of rows and columns.
Note that for n statements, there are 2^n rows,

Truth Table for Two Statement (p, q)

p	q
T	T
T	F
F	T
F	F

3. Elementary Operation of Logic

(i) **Negation:** A statement which is formed by changing the truth value of a given statement by using the word like 'no', 'not' is called negation of given statement. If p is a statement, then negation of p is denoted by $\sim p$.

(ii) **Conjunction:** A compound sentence formed by two simple sentences p and q using connective 'and' is called the conjunction of p and q and it is represented by $p \wedge q$.

(iii) **Disjunction:** A compound sentence formed by two simple sentences p and q using connectives 'or' is called the disjunction of p and q and it is represented by $p \vee q$.

(iv) **Conditional Sentence (Implication):** Two simple sentences p and q connected by the phase, if and then, is called conditional sentence of p and q and it is denoted by $p \rightarrow q$.

(v) **Biconditional Sentence (Bi-implication):** The two simple sentence connected by the phase, 'if and only if' this is called biconditional sentence. It is denoted by the symbol '$\leftrightarrow$'.

Table for Basic Logical Connections

p	q	~p	~q	p ∧ q	p ∨ q	p ⇒ q	p ⇔ q	~(p ∧ q) = ~p ∨ ~q	~(p ⇒ q) = p ∧ ~q	~(p ⇔ q) = (p ∧ ~q) ∨ (~p ∧ q)
T	T	F	F	T	T	T	T	F	F	F
T	F	F	T	F	T	F	F	T	T	T
F	T	T	T	F	T	T	T	T	T	T
F	F	T	F	F	F	T	T	T	T	F

4. Tautology and Contradiction

Two compound statement which are true for every value of their components are called **tautology**.

The compound statements which are false for every value of their components are called **contradiction** (or fallacy).

Truth Table

p	q	p → q	q → p	Tautology (p → q) ∨ (q → p)	Contradiction ~[(p → q) ∨ (q → p)]
T	T	T	T	T	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

5. Laws of Algebra of Statements

(i) Idempotent Laws

(A) $p \vee p = p$ **(B)** $p \wedge p = p$

(ii) Associative Laws

(A) $(p \vee q) \vee r = p \vee (p \vee r)$ **(B)** $(p \wedge q) \wedge r = p \wedge (p \wedge r)$

(iii) Commutative Laws

(A) $p \vee q = q \vee p$ **(B)** $p \wedge q = q \wedge p$

(iv) Distributive Laws

(A) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(B) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(v) De-Morgan's Laws

(A) $\sim(p \vee q) = (\sim p) \wedge (\sim q)$ **(B)** $\sim(p \wedge q) = (\sim p) \vee (\sim q)$

(vi) Identity Laws

(A) $p \wedge F = F$ **(B)** $p \wedge T = p$

(C) $p \vee T = T$ (D) $p \vee F = p$

(vii) Complement Laws

(A) $p \vee (\sim p) = T$ **(B)** $p \wedge (\sim p) = F$

(C) $\sim(\sim p) = p$ (D) $\sim T = F$, $\sim F = T$

6. Important Points to be Remembered

(i) The number of rows of table is depend on the number of statements.

(A) If p is false, then ~p is true. **(B)** If p is true, then ~p is false.

(ii) **(A)** The converse of $p \Rightarrow q$ is $q \Rightarrow p$. **(B)** The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

(iii) The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$.

A statement which is neither a tautology nor a contradiction is a contingency.

7. Validity of Statements

Validity of a statement means checking when the statement is true and when it is not true. This depends upon which of the connectives and quantifiers used in the statement.

(i) Validity of Statement with 'AND'

If p and q are two mathematical statements, then in order to show that the statement '$p \wedge q$' is true, the steps are as follow :

Step I : Show that the statement p is true.

Step II : Show that the statement q is true.

(ii) Validity of Statements with 'OR'

If p and q are mathematical statements, then in order to show that the compound statement 'p or q' is true, as follows.

Case I Assume that p is false, show that q must be true.

or

Case II Assume that q is false, show that q must be true.

(iii) Validity of Statements with 'If-then'

If p and q are two mathematical statements, then in order to show that the compound statement, 'if p then q' is true, the step are as follow

Step I Assume that p is true.

Step II Prove that q must be true i.e., $p \Rightarrow q$.

(iv) Validity of the Statement with 'If and only if'

In order to prove the validity of the statement 'p if and only if q' the steps are as follows

Step I Show that, if p is true, then q is true.

Step II Show that, if q is true, then p is true.