Ellipse

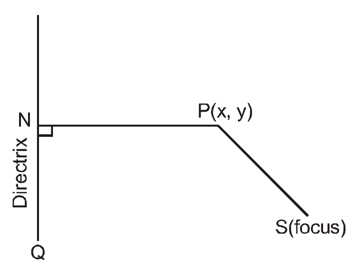
Definitions :

It is locus of a point which moves in such a way that the ratio of its distance from a fixed point and a fixed line (not passes through fixed point and all points and line lies in same plane) is constant (e), which is less than one.

The fixed point is called - **focus**

The fixed line is called **-directrix.**

The constant ratio is called - **eccentricity**, it is denoted by **'e'**.



Let S be the focus, QN be the directrix and P be any point on the ellipse. Then, by definition, $\frac{PS}{PN} = e$ or

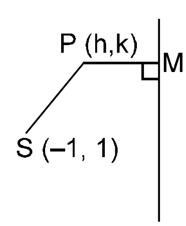
PS = e PN, e < 1, where PN is the length of the perpendicular from P on the directrix QN.

An Alternate Definition An ellipse is the locus of a point that moves in such a way that the sum of its distacnes from two fixed points (called foci) is constant.

Solved Examples

Ex.1 Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $\frac{1}{2}$.

Sol. Let $P \equiv (h, k)$ be moving point,



$e = \frac{PS}{PM} = \frac{1}{2}$

$\Rightarrow (h + 1)^2 + (k - 1)^2$

$= \frac{1}{4}\left(\frac{h - k + 3}{\sqrt{2}}\right)^2$

$\Rightarrow$ locus of P(h, k) is

$8\ \{x^2 + y^2 + 2x - 2y + 2\}$

$= (x^2 + y^2 - 2xy + 6x - 6y + 9)$

$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$**.**

Note :

The general equation of a conic with focus (p, q) & directrix $\ell x + my + n = 0$ is:

$(\ell^2 + m^2)\ [(x - p)^2 + (y - q)^2] = e^2(\ell x + my + n)^2$

$\equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

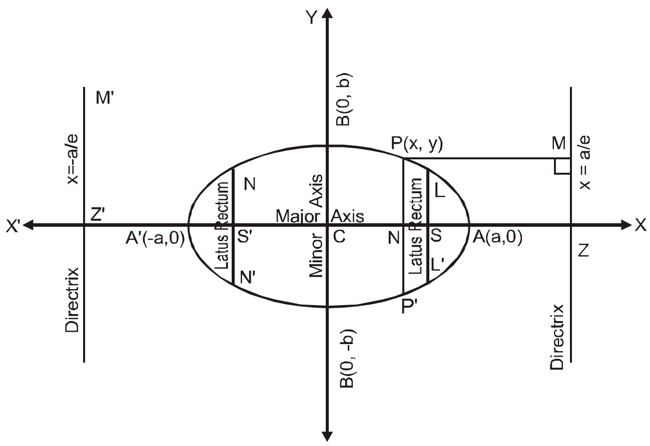
represent ellipse if $0 < e < 1$; $\Delta \neq 0$, $h^2 < ab$

EQUATION OF AN ELLIPSE IN STANDARD FORM

The **Standard form** of the equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b),$$

where a and b are constants.



TERMS RELATED TO AN ELLIPSE

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$, has been shown in the figure given above.

Symmetry

(a) On replacing y by –y, the above equation remains unchanged. So, the curve is symmetrical about x-axis.

(b) On replacing x by –x, the above equation remains unchanged. So, the curve is symmetrical about y-axis

Eccentricity: $e=\sqrt{1-\frac{b^2}{a^2}}$, $(0<e<1)$

Focii : $S\equiv(ae,0)$ & $S'\equiv(-ae,0)$.

Equations of Directrices : $x=\frac{a}{e}$ & $x=-\frac{a}{e}$.

Major Axis : The line segment A′A in which the focii S′ & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).

Minor Axis : The y–axis intersects the ellipse in the points $B'\equiv(0,-b)$ & $B\equiv(0,b)$. The line segment B′B is of length 2b (b < a) is called the minor axis of the ellipse.

Principal Axis : The major & minor axes together are called principal axis of the ellipse.

Vertices : Point of intersection of ellipse with major axis. $A'\equiv(-a,0)$ & $A\equiv(a,0)$.

Focal Chord : A chord which passes through a focus is called a focal chord.

Double Ordinate : A chord perpendicular to the major axis is called a double ordinate.

Latus Rectum : The focal chord perpendicular to the major axis is called the latus rectum.

Length of latus rectum **(LL′)**

$$=\frac{2b^2}{a}=\frac{(\text{minor axis})^2}{\text{major axis}}=2a\left(1-e^2\right)$$

= 2 e (distance from focus to the corresponding directrix)

Centre : The point which bisects every chord of the conic drawn through it, is called the centre of the conic. $C\equiv(0,0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Note :

(i) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and nothing is mentioned then the rule is to assume that a > b.

(ii) If b > a is given, then the y–axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.

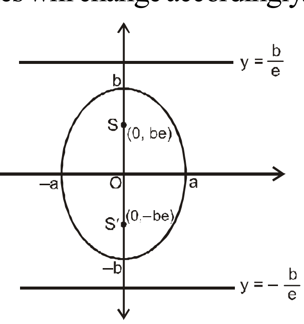
Equation : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Foci $(0,\pm be)$

Directrices : $y=\pm\frac{b}{e}$

$a^2=b^2(1-e^2)$, a < b.

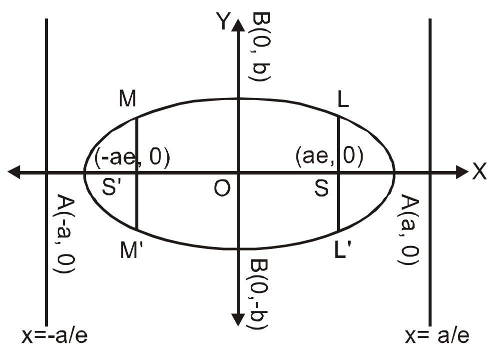
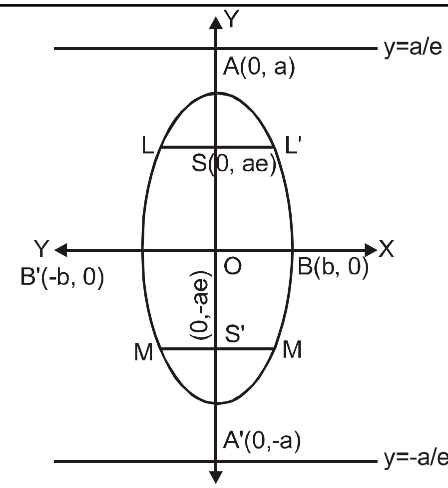
$\Rightarrow e=\sqrt{1-\frac{a^2}{b^2}}$



Vertices $(0,\pm b)$; L.R. $y=\pm be$

ℓ (L·R.) $=\frac{2a^2}{b}$, centre : (0, 0)

TWO STANDARD FORMS OF THE ELLIPSE

Standard equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$ (Horizontal Form of an Ellipse)	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1\;(a>b)$ (Vertical Form of an ellipse)
Shape of the ellipse		
Centre	(0, 0)	(0, 0)
Equation of major axis	y = 0	x = 0
Equation of minor axis	x = 0	y = 0
Length of major axis	2a	2a
Length of minor axis	2b	2b
Foci	(± ae, 0)	(0, ± ae)
Vertices	(± a, 0)	(0, ± a)
Equation of directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
Eccentricity	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{a^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Ends of latus rectum	$\left(\pm ae,\pm\frac{b^2}{a}\right)$	$\left(\pm\frac{b^2}{a},\pm ae\right)$
Parametric coordinates	$(a\cos\theta,\ b\sin\theta)$	$(a\cos\theta,\ b\sin\theta)$
Focal radius	$P = a - ex_1$ and $S'P = a + ex_1$	$SP = a - ey_1$ and $S'P = a + ey_1$.
Sumoffocalradii SP+S'P=	2a	2a
Distance between foci	2ae	2ae
Distance b/w directrices	$\frac{2a}{e}$	$\frac{2a}{e}$
Tangents at the vertices	x = ± a	y = ± a

Solved Examples

Ex.2 Find the centre, the length of the axes, eccentricity and the foci of the ellipse.

$12x^2 + 4y^2 + 24x - 16y + 25 = 0$

Sol. The given equation can be written in the form

$12(x + 1)^2 + 4(y - 2)^2 = 3$ or

$$\frac{(x+1)^2}{1/4} + \frac{(y-2)^2}{3/4} = 1 \quad ....(1)$$

Co-ordinates of centre of the ellipse are given by

$x + 1 = 0$ and $y - 2 = 0$

hence centre of the ellipse is $(-1, 2)$

If a and b be the lengths of the semi major and semi minor axes, then $a^2 = 3/4$, $b^2 = 1/4$

$\therefore$ Length of major axis $= 2a = \sqrt{3}$,

Length of minor axis $= 2b = 1$ $\therefore$ $a = \frac{\sqrt{3}}{2}, b = \frac{1}{2}$

Since $b^2 = a^2(1 - e^2)$ $\therefore$ $1/4 = 3/4\,(1 - e^2)$

$\Rightarrow$ $e = \sqrt{2/3}$ $\qquad \therefore ae = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{\sqrt{2}}$,

Co-ordinates of foci are given by

$x + 1 = 0, y - 2 = \pm ae$

Thus foci are $\left(-1, 2 \pm \frac{1}{\sqrt{2}}\right)$

Ex.3 Find the equation of the ellipse having its centre at the point (2, –3) one focus at (3, –3) and one vertex at (4, –3).

Sol. $C \equiv (2, -3)$, $S \equiv (3, -3)$ and $A \equiv (4, -3)$

Now $CA = \sqrt{(4-2)^2 + (-3+3)^2} = 2$

$\therefore$ $a = 2$

Again $CA = \sqrt{(3-2)^2 + (-3+3)^2} = 1$

$\therefore ae = 1$ $\quad \therefore e = \frac{1}{a} = \frac{1}{2}$

Let the directrix cut the major-axis at Q.

Then $\frac{AS}{AQ} = e = \frac{1}{2}$

If $Q \equiv (\alpha, \beta)$, then SA : AQ = e : 1 = 1 : 2

$\therefore$ $A = \left(\frac{\alpha+6}{3}, \frac{\beta-6}{3}\right) = (4, -3)$

$\Rightarrow$ $\frac{\alpha+6}{3} = 4$, $\frac{\beta-6}{3} = -3$ $\quad \therefore$ $\alpha = 6, \beta = -3$

Slope of CA = 0, therefore directrix will be parallel to y-axis.

Since directrix is parallel to y-axis and it passes through point Q(6, –3)

$\therefore$ equation of directrix will be $x = 6$

Let P(x, y) is a point on ellipse. then

$$e = \frac{1}{2} = \frac{PS}{PN} = \frac{\sqrt{(x-3)^2 + (y+3)^2}}{\left|x - 6/\sqrt{1^2}\right|}$$

$\therefore$ $\frac{1}{4} = \frac{(x-3)^2 + (y+3)^2}{(x-6)^2}$; or

$(x - 6)^2 = 4\,[(x - 3)^2 + (y + 3)^2]$

or $x^2 - 12x + 36 = 4\,[x^2 - 6x + 9 + y^2 + 6y + 9]$

or $3x^2 + 4y^2 - 12x + 24y + 36 = 0$

Ex.4 Find the equation to the ellipse whose centre is origin, axes are the axes of co-ordinate and passes through the points (2, 2) and (3, 1).

Sol. Let the equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through the points (2, 2) and (3, 1)

$\therefore$ $\frac{4}{a^2} + \frac{4}{b^2} = 1$ ....(i)

and $\frac{9}{a^2} + \frac{1}{b^2} = 1$ ....(ii)

from (i) – 4 (ii), we get

$\frac{4-36}{a^2} = 1 - 4$ $\qquad \Rightarrow \qquad a^2 = \frac{32}{3}$

from (i), we get

$\frac{1}{b^2} = \frac{1}{4} - \frac{3}{32} = \frac{8-3}{32}$ $\qquad b^2 = \frac{32}{5}$

$\therefore$ Ellipse is $3x^2 + 5y^2 = 32$

Ex.5 Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\frac{1}{3}$

Sol. Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes through centre is minor axis which is y-axis.

Let equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\because$ ae = 4 and $e = \frac{1}{3}$ (Given)

$\therefore$ a = 12 and $b^2 = a^2(1 - e^2)$

$\Rightarrow b^2 = 144\left(1 - \frac{1}{9}\right)$

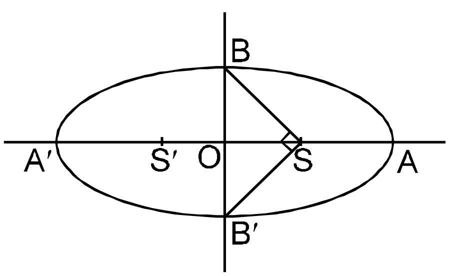
$b^2 = 16 \times 8$

$b = 8\sqrt{2}$

Equation of ellipse is $\frac{x^2}{144} + \frac{y^2}{128} = 1$

Ex.6 If minor-axis of ellipse subtend a right angle at its focus then find the eccentricity of ellipse.

Sol. Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)



$\because \angle BSB' = \frac{\pi}{2}$

and OB = OB′

$\therefore \angle BSO = \frac{\pi}{4}$

$\Rightarrow$ OS = OB $\Rightarrow$ ae = b

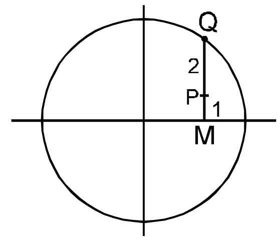
$\Rightarrow e^2 = \frac{b^2}{a^2} = 1 - e^2 \Rightarrow e = \frac{1}{\sqrt{2}}$

Ex.7 From a point Q on the circle $x^2 + y^2 = a^2$, perpendicular QM are drawn to x-axis, find the locus of point 'P' dividing QM in ratio 2 : 1.

Sol. Let $Q \equiv (a\cos\theta, a\sin\theta)$

$M \equiv (a\cos\theta, 0)$

Let $P \equiv (h, k)$



$\therefore h = a\cos\theta,\ k = \frac{a\sin\theta}{3}$

$\therefore \left(\frac{3k}{a}\right)^2 + \left(\frac{h}{a}\right)^2 = 1$

$\Rightarrow$ Locus of P is $\frac{x^2}{a^2} + \frac{y^2}{(a/3)^2} = 1$

Ex.8 Find the equation of axes, directrix, co-ordinate of focii, centre, vertices, length of latus - rectum and eccentricity of an ellipse $\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$.

Sol. Let x – 3 = X, y – 2 = Y, so equation of ellipse becomes as $\frac{X^2}{5^2} + \frac{Y^2}{4^2} = 1$.

equation of major axis is Y = 0 $\Rightarrow$ y = 2.

equation of minor axis is X = 0 $\Rightarrow$ x = 3.

centre (X = 0, Y = 0) $\Rightarrow$ x = 3, y = 2

$C \equiv (3, 2)$

Length of semi-major axis a = 5

Length of major axis 2a = 10

Length of semi-minor axis b = 4

Length of minor axis = 2b = 8.

Let 'e' be eccentricity

$\therefore b^2 = a^2(1 - e^2)$

$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{25 - 16}{25}} = \frac{3}{5}$.

Length of latus rectum = LL′ = $\frac{2b^2}{a} = \frac{2 \times 16}{5} = \frac{32}{5}$

Co-ordinates focii are X = ± ae, Y = 0

$\Rightarrow S \equiv (X = 3, Y = 0)$ & $S' \equiv (X = -3, Y = 0)$

$\Rightarrow S \equiv (6, 2)$ & $S' \equiv (0, 2)$

Co-ordinate of vertices

Extremities of major axis

$A \equiv (X = a, Y = 0)$ & $A' \equiv (X = -a, Y = 0)$

$\Rightarrow A \equiv (x = 8, y = 2)$ & $A' = (x = -2, 2)$

$A \equiv (8, 2)$ & $A' \equiv (-2, 2)$

Extremities of minor axis

$B \equiv (X = 0, Y = b)$ & $B' \equiv (X = 0, Y = -b)$

$B \equiv (x = 3, y = 6)$ & $B' \equiv (x = 3, y = -2)$

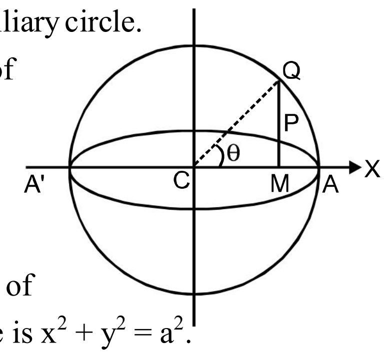
$B \equiv (3, 6)$ & $B' \equiv (3, -2)$

Equation of directrix $X = \pm \frac{a}{e}$

$x - 3 = \pm \frac{25}{3} \Rightarrow x = \frac{34}{3}$ & $x = -\frac{16}{3}$

Auxillary Circle

The circle drawn on major axis AA' as diameter is known as the Auciliary circle.



Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Then the equation of its auxiliary circle is $x^2 + y^2 = a^2$.

Let Q be a point on auxiliary circle so that QM, perpendicular to major axis meets the ellipse at P. The points P and Q are called as corresponding point on the ellipse and auxiliary circle respectively.

The angle θ is known as eccentric angle of the point P on the ellipse.

It may be noted that the CQ and not CP is inclined at θ with x-axis.

Note that :

$$\frac{\ell(PM)}{\ell(QM)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$$

NOTE :

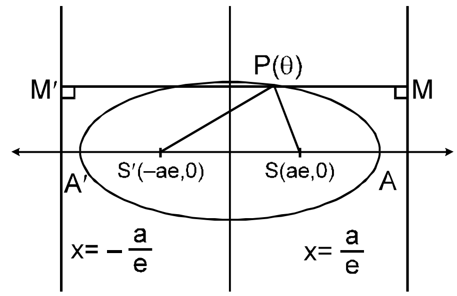
If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

Solved Examples

Ex.9 Find the focal distance of a point $P(\theta)$ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a > b)$$

Sol. Let 'e' be the eccentricity of ellipse.



$\therefore$ $PS = e \,.\, PM = e\left(\frac{a}{e} - a\cos\theta\right)$

$PS = (a - a\,e\cos\theta)$

and $PS' = e.\, PM' = e\left(a\cos\theta + \frac{a}{e}\right)$

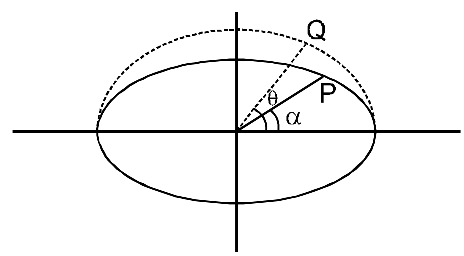
$PS' = a + ae\cos\theta$

$\therefore$ focal distance are $(a \pm ae\cos\theta)$

Note : $PS + PS' = 2a$ $\quad PS + PS' = AA'$

Ex.10 Find the distance from centre of the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose radius makes angle α with x-axis.

Sol. Let $P \equiv (a\cos\theta, b\sin\theta)$



$\therefore m_{(op)} = \frac{b}{a}\tan\theta = \tan\alpha \Rightarrow \tan\theta = \frac{a}{b}\tan\alpha$

$$OP = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \sqrt{\frac{a^2 + b^2\tan^2\theta}{\sec^2\theta}}$$

$$= \sqrt{\frac{a^2 + b^2\tan^2\theta}{1 + \tan^2\theta}} = \sqrt{\frac{a^2 + b^2 \times \frac{a^2}{b^2}\tan^2\alpha}{1 + \frac{a^2}{b^2}\tan^2\alpha}}$$

$$OP = \frac{ab}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}$$

PARAMETRIC EQUATION OF THE ELLIPSE

The coordinates x = a cos θ and y = b sin θ satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

for all real values of θ. Thus, x = a cos θ, y = b sin θ are the parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameter $0 \le\ < 2\pi$.

Hence the coordinates of any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

may be taken as (a cos θ, b sin θ). This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point (a cos θ, b sin θ) on the ellipse.

Equation of Chord

The equation of the chord joining the points P ≡ (acos θ_1, bsin θ_1) and Q ≡ (acos θ_2, bsin θ_2) is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right).$$

Solved Examples

Ex.11 Write the equation of chord of an ellipse

$\frac{x^2}{25}+\frac{y^2}{16}=1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

Sol. Equation of chord is $\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}+\frac{y}{4}.$

$$\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2}=\cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$$

$$\frac{x}{5}.\cos\left(\frac{3\pi}{4}\right)+\frac{y}{4}.\sin\left(\frac{3\pi}{4}\right)=0$$

$$-\frac{x}{5}+\frac{y}{4}=0 \quad\Rightarrow\quad 4x=5y$$

Ex.12 If P(α) and P(β) are extremities of a focal chord of ellipse then prove that its eccentricity

$$e=\left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right|.$$

Sol. Let the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

∴ equation of chord is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

Since above chord is focal chord,

∴ it passes through focus (ae, 0) or (– ae, 0)

$$\therefore\ \pm e\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\therefore\ e=\left|\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right| \qquad \textbf{Ans.}$$

Note :

$$\because\ \pm e=\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \quad\Rightarrow\quad \pm e=\frac{1+\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}$$

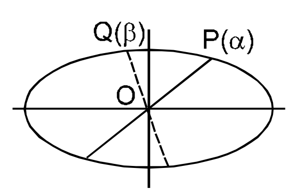
Applying componendo and dividendo

$$\frac{1\pm e}{\pm e-1}=\frac{2}{2\tan\frac{\alpha}{2}.\tan\frac{\beta}{2}}$$

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1+e}{e-1}\ \text{or}\ \frac{e-1}{1+e}$$

Ex.13 Find the angle between two diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Whose extremities have eccentricity angle α and $\beta=\alpha+\frac{\pi}{2}$.

Sol. Let ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Slope of OP = $m_1 = \frac{b\sin\alpha}{a\cos\alpha} = \frac{b}{a}\tan\alpha$

Slope of OQ = $m_2 = \frac{b\sin\beta}{a\cos\beta} = -\frac{b}{a}\cot\alpha$

given $\beta = \alpha + \frac{\pi}{2}$

$$\therefore\ \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{b}{a}(\tan\alpha + \cot\alpha)}{1 - \frac{b^2}{a^2}}\right|$$

$$= \left|\frac{2ab}{(a^2 - b^2)\sin 2\alpha}\right|$$

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE

The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ according as

$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0, = 0$ or < 0.

Solved Examples

Ex.14 Check whether the point P(3, 2) lies inside or outside of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Sol. $S_1 \equiv \frac{9}{25} + \frac{4}{16} - 1 = \frac{9}{25} + \frac{1}{4} - 1 < 0$

$\therefore$ Point $P \equiv (3, 2)$ lies inside the ellipse.

Ex.15 Find the set of value(s) of 'α' for which the point $P(\alpha, -\alpha)$ lies inside the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Sol. If $P(\alpha, -\alpha)$ lies inside the ellipse

$\therefore\ S_1 < 0$

$\Rightarrow \frac{\alpha^2}{16} + \frac{\alpha^2}{9} - 1 < 0$

$\Rightarrow \frac{25}{144} \cdot \alpha^2 < 1 \qquad \Rightarrow \qquad \alpha^2 < \frac{144}{25}$

$\therefore\ \alpha \in \left(-\frac{12}{5}, \frac{12}{5}\right)$.

CONDITION OF TANGENCY AND POINT OF CONTACT

The condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are

$$\left(\pm \frac{a^2 m}{\sqrt{a^2m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$$

Note

* $x\cos a + y\sin a = p$ is a tangent if $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.

* $lx + my + n = 0$ is a tangent if $n^2 = a^2l^2 + b^2m^2$.

Solved Examples

Ex.16 Find the set of value(s) of 'λ' for which the line $3x - 4y + \lambda = 0$ intersect the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at two distinct points.

Sol. Solving given line with ellipse,

we get $\frac{(4y - \lambda)^2}{9 \times 16} + \frac{y^2}{9} = 1$

$\frac{2y^2}{9} - \frac{y\lambda}{18} + \frac{\lambda^2}{144} - 1 = 0$

Since, line intersect the parabola at two distinct points,

$\therefore$ roots of above equation are real & distinct

$\therefore\ D > 0$

$\Rightarrow \frac{\lambda^2}{(18)^2} - \frac{8}{9} \cdot \left(\frac{\lambda^2}{144} - 1\right) > 0$

$\Rightarrow -12\sqrt{2} < \lambda < 12\sqrt{2}$

Tangents :

(a) Slope form: $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m.

(b) Point form : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

(c) Parametric form: $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$.

Note :

(i) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents at the point α & β is, $\left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(iii) The eccentric angles of the points of contact of two parallel tangents differ by π.

Solved Examples

Ex.17 Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Slope of tangent = $m = \frac{1}{2}$

Given ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Equation of tangent whose slope is 'm' is

$y = mx \pm \sqrt{4m^2 + 3}$

$\because$ $m = \frac{1}{2}$

$\therefore$ $y = \frac{1}{2}x \pm \sqrt{1+3}$

$2y = x \pm 4$

Ex.18 For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. $\because$ Equation of ellipse is

$9x^2 + 16y^2 = 144$ or $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

then we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore$ $m = 1$ and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse

$9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$\Rightarrow \lambda^2 = 16 \times 1^2 + 9$ $\Rightarrow \lambda^2 = 25$ $\therefore$ $\lambda = \pm 5$

Ex.19 Find the equation of tangent to the ellipse $4x^2 + 9y^2 = 36$ at the point $(3, -2)$.

Sol. We have $4x^2 + 9y^2 = 36$ $\Rightarrow$ $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

where $a^2 = 9$ and $b^2 = 4$.

We, know that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

So, the eqation of the tangent to the given ellipse at $(3, -2)$ is

$\frac{3x}{9} - \frac{2y}{4} = 1$ i.e., $\frac{x}{3} - \frac{y}{2} = 1$

Ex.20 Find the equations of the tangetns to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Let m be the slope of the tangent, since the tangent is perpendicualr to the line $y + 2x = 4$.

$\therefore$ $mx - 2 = -1$ $\Rightarrow$ $m = \frac{1}{2}$

Since $3x^2 + 4y^2 = 12$ or $\frac{x^2}{4} + \frac{y^2}{3} = 1$

Comparing this with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ $a^2 = 4$ and $b^2 = 3$

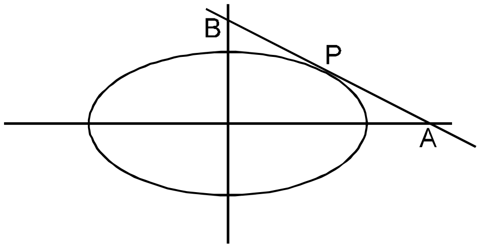
So the equation of the tangent are

$y = \frac{1}{2}x \pm \sqrt{4x\frac{1}{2} + 3}$

$\Rightarrow$ $y = \frac{1}{2}x \pm 2$ or $x - 2y \pm 4 = 0$

Ex.21 A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant and meets the co-ordinate axes in A and B respectively. If P divides AB in the ratio 3 : 1, find the equation of the tangent.

Sol. Let P ≡ (a cosθ, b sinθ)



$\therefore$ equation of tangent is

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

A ≡ (a secθ, 0) B ≡ (0, b cosecθ)

$\because$ P divide AB internally in the ratio 3 : 1

$\therefore$ $a\cos\theta = \frac{a\sec\theta}{4}$

$\Rightarrow \cos^2\theta = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$ and

$b\sin\theta = \frac{3b\,\text{cosec}\theta}{4} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ tangent is $\frac{x}{2a} + \frac{\sqrt{3}y}{2b} = 1$

$\Rightarrow bx + \sqrt{3}\,ay = 2ab$

Ex.22 Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angle differ by a constant α is an ellipse.

Sol. Let P (h, k) be the point of intersection of tangents at A(θ) and B(β) to the ellipse.

$$\therefore h = \frac{a\cos\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)} \;\&\; k = \frac{b\sin\left(\frac{\theta+\beta}{2}\right)}{\cos\left(\frac{\theta-\beta}{2}\right)}$$

$$\Rightarrow \left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 = \sec^2\left(\frac{\theta-\beta}{2}\right)$$

but given that $\theta - \beta = \alpha$

$$\therefore \text{locus is } \frac{x^2}{a^2\sec^2\left(\frac{\alpha}{2}\right)} + \frac{y^2}{b^2\sec^2\left(\frac{\alpha}{2}\right)} = 1$$

Ex.23 Find the locus of foot of perpendicular drawn from centre to any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Let P(h, k) be the foot of perpendicular to a tangent $y = mx + \sqrt{a^2m^2 + b^2}$ ....(i)

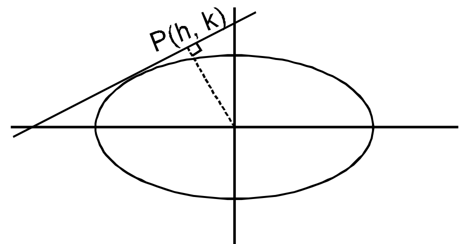
from centre

$\therefore \frac{k}{h}.m = -1 \quad \Rightarrow \quad m = -\frac{h}{k}$ ....(ii)

$\because$ P(h, k) lies on tangent

$\therefore k = mh + \sqrt{a^2m^2 + b^2}$ ....(iii)

from equation (ii) & (iii), we get



$$\left(k + \frac{h^2}{k}\right)^2 = \frac{a^2h^2}{k^2} + b^2$$

$\Rightarrow$ locus is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

EQUATION OF NORMAL IN DIFFERENT FORMS

(i) Point Form

The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

(ii) Parametric Form

The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (acos θ, bsin θ) is $ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$.

or $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.

(iii) Slope Form

The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of slope 'm' is $y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$

Notes :

* The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right)$$

* **Condition for normality** The line = mx + c is normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ if } c^2=\frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}$$

* **Number of Normals**

In general, four normals can be drawn to an ellipse from a point in its plane i.e., there are four points on the ellipse, the normals at which it will pass through a given point. These four points are called the co-normal points.

* if $\alpha, \beta, \gamma, \delta$ are the eccentric angles of the four points on the ellipse such that the normals at these points are concurrent, then $(\alpha+\beta+\gamma+\delta)$ is an odd multiple of π.
* If α, β, γ are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the normals at which are concurrent, then

$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$

Solved Examples

Ex.24 Find the condition that the line lx + my = n may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Sol. Equation of normal to the given ellipse at

(a cos θ, b sin θ) is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ...(1)

If the line lx + my = n is also normal to the ellipse then there must be a value of θ for which line (1) and line lx + my = n are identical. For that value of θ we have

$$\frac{1}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{-\left(\frac{b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)}$$ or

$$\frac{\ell}{a}\cos\theta=\frac{an}{\ell(a^2-b^2)}$$ ....(3)

and $\sin\theta=\frac{-bn}{m(a^2-b^2)}$ ...(4)

Squaring and adding (3) and (4), we get

$1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{\ell^2}+\frac{b^2}{m^2}\right)$ which is the required condition.

Ex.25 If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e^4+e^2-1=0$ or $e^2=\frac{\sqrt{5}-1}{2}$

Sol. The co-ordinates of an end of the latus-rectum are $(ae, b^2/a)$. The equation of notrmal at $P(ae, b^2/a)$ is

$$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2 \quad \text{or} \quad \frac{ax}{e}-ay=a^2-b^2$$

If passes through one extremity of the minor axis whose co-ordinates are (0, –b)

$\therefore\ 0+ab=a^2-b^2 \quad \Rightarrow \quad (a^2b^2)=(a^2-b^2)^2$

$\Rightarrow\ a^2.a^2(1-e^2)=(a^2e^2)^2$

$\Rightarrow\ 1-e^2=e^4 \quad \Rightarrow \quad e^4+e^2-1=0$

$\Rightarrow\ (e^2)^2+e^2-1=0$

$\therefore\ e^2=\frac{-1\pm\sqrt{1+4}}{2} \quad \Rightarrow \quad e^2=\frac{\sqrt{5}-1}{2}$

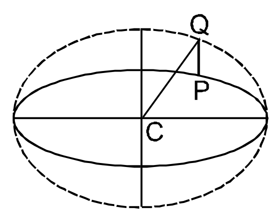
(taking positive sign)

Ex.26 P and Q are corresponding points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

Sol. Let P ≡ (acos θ, b sinθ)

∴ Q ≡ (a cosθ, a sinθ)

Equation of normal at P is



(a secθ) x – (b cosec θ) y = $a^2 - b^2$ ....(i)

equation of CQ is y = tanθ . x ....(ii)

Solving equation (i) & (ii), we get (a – b) x = $(a^2 - b^2)$ cosθ

x = (a + b) cosθ, & y = (a + b) sinθ

∴ R ≡ ((a + b) cosθ, (a + b) sinθ)

∴ CR = a + b

Ex.27 Find the shortest distance between the line $x + y = 10$ and the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Sol. Shortest distance occurs between two non-intersecting curve always along common normal.

Let 'P' be a point on ellipse and Q is a point on given line for which PQ is common normal.

$\therefore$ Tangent at 'P' is parallel to given line

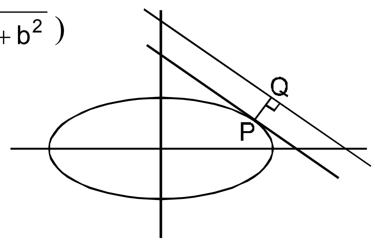
$\therefore$ Equation of tangent parallel to given line is

$(y = mx \pm \sqrt{a^2m^2 + b^2})$

$y = -x \pm 5$

$\Rightarrow x + y + 5 = 0$

or $x + y - 5 = 0$



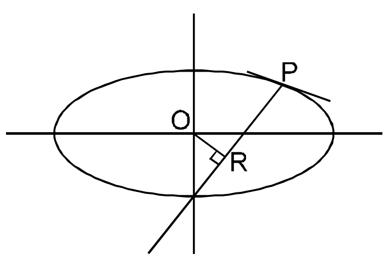
$\therefore$ minimum distance = distance between $x + y - 10 = 0$ & $x + y - 5 = 0$

$\Rightarrow$ shortest distance $= \frac{|10-5|}{\sqrt{1+1}} = \frac{5}{\sqrt{2}}$

Ex.28 Prove that, in an ellipse, the distance between the centre and any normal does not exceed the difference between the semi-axes of the ellipse.

Sol. Let the equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of normal at P (θ) is $(a\sec\theta)x - (b\,\mathrm{cosec}\,\theta)y - a^2 + b^2 = 0$ distance of normal from centre



$= OR = \frac{|a^2 - b^2|}{\sqrt{a^2 + b^2 + (a\tan\theta)^2 + (b\cot\theta)^2}}$

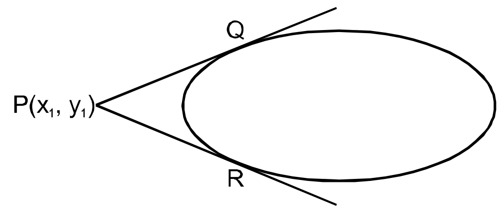
$= \frac{|a^2 - b^2|}{\sqrt{(a+b)^2 + (a\tan\theta - b\cot\theta)^2}}$

$\because (a + b)^2 + (a\tan\theta - b\cot\theta)^2 \ge (a + b)^2$ or

$\le \frac{|a^2 - b^2|}{\sqrt{(a+b)^2}}$ $\qquad |OR| \le (a - b)$

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point



$P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $SS_1 = T^2$

where

$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,\ S_1 \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$

and $\qquad T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

Solved Examples

Ex.29 How many real tangents can be drawn from the point (4, 3) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

Sol. Given point $P \equiv (4, 3)$

ellipse $S \equiv \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$

$\because S_1 \equiv \frac{16}{16} + \frac{9}{9} - 1 = 1 > 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the ellipse.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is $SS_1 = T^2$

$\Rightarrow \left(\frac{x^2}{16} + \frac{y^2}{9} - 1\right) . 1 = \left(\frac{4x}{16} + \frac{3y}{9} - 1\right)^2$

$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} - 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 + \frac{xy}{6} - \frac{x}{2} - \frac{2y}{3}$

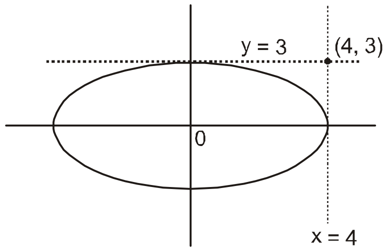
$\Rightarrow -xy + 3x + 4y - 12 = 0$

$\Rightarrow (4 - x)(y - 3) = 0$

$\Rightarrow x = 4$ & $y = 3$ and angle between them $= \frac{\pi}{2}$

Alternative

By direct observation



x = 4, y = 3 are tangents.

Solved Examples

Ex.30 Find the locus of point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol. Let P(h, k) be the point of intersection of two perpendicular tangents equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)=\left(\frac{hx}{a^2}+\frac{ky}{b^2}-1\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{y^2}{b^2}\left(\frac{h^2}{a^2}-1\right)+........=0 \quad ....(i)$$

Since equation (i) represents two perpendicular lines

$$\therefore \frac{1}{a^2}\left(\frac{k^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{h^2}{a^2}-1\right)=0$$

$\Rightarrow k^2 - b^2 + h^2 - a^2 = 0$

$\Rightarrow$ locus is $x^2 + y^2 = a^2 + b^2$

Director Circle :

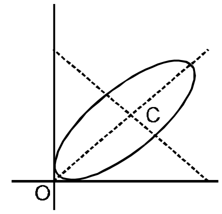
Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axes.

Solved Examples

Ex.31 An ellipse slides between two perpendicular lines. Show that the locus of its centre is a circle.

Sol. Let length of semi-major and semi-minor axis are 'a' and 'b' and centre is $C \equiv (h, k)$

Since ellipse slides between two perpendicular lines, therefore point of intersection of two perpendicular tangents lies on director circle.



Let us consider two perpendicular lines as x & y axes

$\therefore$ point of intersection is origin $O \equiv (0, 0)$

$\therefore$ OC = radius of director circle

$\therefore \sqrt{h^2+k^2} = \sqrt{a^2+b^2}$

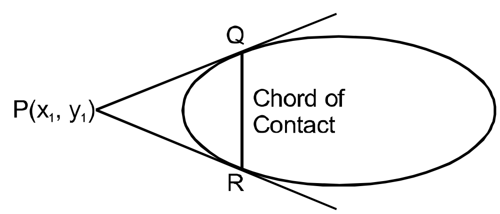
$\Rightarrow$ locus of $C \equiv (h, k)$ is

$\Rightarrow x^2 + y^2 = a^2 + b^2$ which is a circle

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

T = 0, where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.



Solved Examples

Ex.32 If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Sol. Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2}+\frac{yk}{b^2}$ = 1 ....(i)

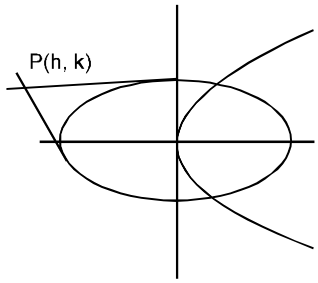
which touches the parabola equation of tangent to parabola $y^2 = 4ax$

$y = mx + \frac{a}{m}$

$\Rightarrow mx - y = -\dfrac{a}{m}$ ....(ii)

equation (i) & (ii) as must be same

$\therefore \dfrac{m}{\left(\dfrac{h}{a^2}\right)} = \dfrac{-1}{\left(\dfrac{k}{b^2}\right)} = \dfrac{-\dfrac{a}{m}}{1}$

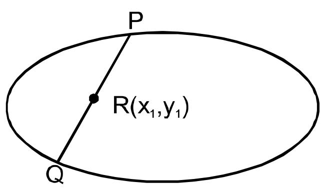


$\Rightarrow m = -\dfrac{h}{k}\dfrac{b^2}{a^2}$ & $m = \dfrac{ak}{b^2}$

$\therefore -\dfrac{hb^2}{ka^2} = \dfrac{ak}{b^2}$ $\Rightarrow$ locus of P is $y^2 = -\dfrac{b^4}{a^3} \cdot x$

CHORD WITH A GIVEN MID POINT

The equation of the chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $P(x_1, y_1)$ as its middle point is given by $T = S_1$



where

$T \equiv \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} - 1$ and $S_1 \equiv \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1$.

Solved Examples

Ex.33 Find the locus of the mid - point of focal chords of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

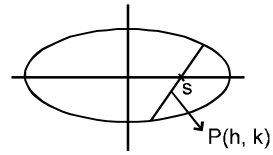
Sol. Let $P \equiv (h, k)$ be the mid-point

$\therefore$ equation of chord whose mid-point is given $\dfrac{xh}{a^2} + \dfrac{yk}{b^2} - 1 = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ it passes through focus, either (ae, 0) or (–ae, 0)

If it passes through (ae, 0)



$\therefore$ locus is $\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

If it passes through (–ae, 0)

$\therefore$ locus is $-\dfrac{ex}{a} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

Ex.34 Find the condition on 'a' and 'b' for which two distinct chords of the ellipse $\dfrac{x^2}{2a^2} + \dfrac{y^2}{2b^2} = 1$ passing through (a, – b) are bisected by the line x + y = b.

Sol. Let the line x + y = b bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ equation of chord whose mid-point is $P(\alpha, b - \alpha)$

$\dfrac{x\alpha}{2a^2} + \dfrac{y(b-\alpha)}{2b^2} = \dfrac{\alpha^2}{2a^2} + \dfrac{(b-\alpha)^2}{2b^2}$

Since it passes through (a, –b)

$\therefore \dfrac{\alpha}{2a} - \dfrac{(b-\alpha)}{2b} = \dfrac{\alpha^2}{2a^2} + \dfrac{(b-\alpha)^2}{2b^2}$

$\Rightarrow \left(\dfrac{1}{a} + \dfrac{1}{b}\right)\alpha - 1 = \alpha^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) - \dfrac{2}{b}\alpha + 1$

$\Rightarrow \alpha^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) - \left(\dfrac{3}{b} + \dfrac{1}{a}\right)\alpha + 2 = 0$

since line bisect two chord

$\therefore$ above quadratic equation in α must have two distinct real roots

$\therefore \left(\dfrac{3}{b} + \dfrac{1}{a}\right)^2 - 4\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) . 2 > 0$

$\Rightarrow \dfrac{9}{b^2} + \dfrac{1}{a^2} + \dfrac{6}{ab} - \dfrac{8}{a^2} - \dfrac{8}{b^2} > 0$

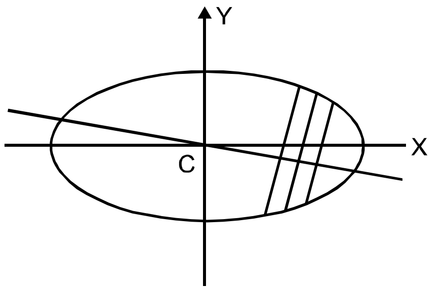
$\Rightarrow \dfrac{1}{b^2} - \dfrac{7}{a^2} + \dfrac{6}{ab} > 0$

$\Rightarrow a^2 - 7b^2 + 6ab > 0$

$\Rightarrow a^2 > 7b^2 - 6ab$ which is the required condition.

DIAMETER OF AN ELLIPSE

The locus of the middle points of a system of a parallel chords of an ellipse is called a diameter of the ellipse.



The equation of the diameter bisecting chords of slope m of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=\frac{b^2}{a^2m}x.$

Note :

Diameter of an ellipse always passes through its centre. Thus a diameter of an ellipse is its chord passing through its centre.

CONJUGATE DIAMETERS

Two diameters of an ellipse are said to be conjugate diameters if each bisects the chord parallel to the other.

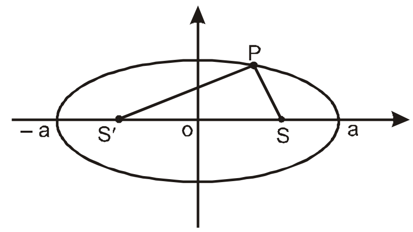
Note :

* Major and minor axes of an ellipse is also a pair of conjugate diameters.
* If m_1 and m_2 be the slopes of the conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1m_2=\frac{-b^2}{a^2}$.
* The eccentric angles of the ends of a pair of conjugate diameters differ by a right angle. i.e., if PCP' and QCQ' is a pair of conjugate diameters and if eccentric angle of P is θ, then eccentric angles of Q, P', Q' (proceeding in anticlockwise direction) will be $\theta+\frac{\pi}{2}$, $\theta+\pi$ and $\theta+\frac{3\pi}{2}$ respectively.

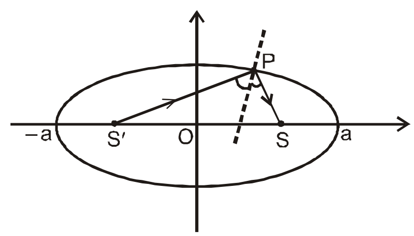
Important Highlights :

Refering to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(1) If P be any point on the ellipse with S & S′ as its foci then $\ell(SP)+\ell(S'P)=2a$.

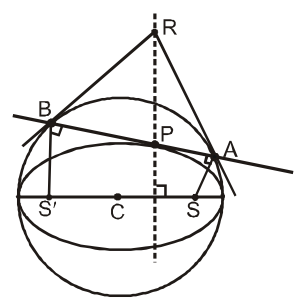


(2) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.

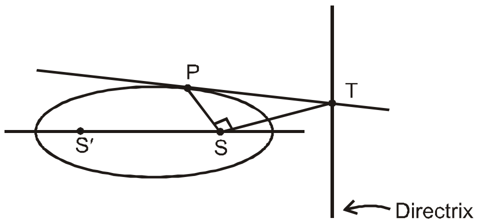


(3) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b² and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of

intersection is a similiar ellipse as that of the original one.



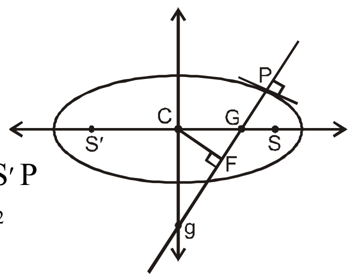
(4) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.



(5) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) PF. PG = b^2

(ii) PF. Pg = a^2



(iii) PG. Pg = SP. S′P

(iv) CG. CT = CS^2

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(6) The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

(7) If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,

(i) T t. PY = $a^2 - b^2$ and

(ii) least value of T t is a + b.