WAVES

1. WAVES :

Waves is distributed energy or distributed "disturbance (force)"

- **Following points regarding waves :**

1. The disturbance (force) is transmitted from one point to another.
2. The energy is transmitted from one point to another.
3. The energy or distrubance passes in the form of wave without any net displacement of medium.
4. The oscillatory motion of preceding particle is imparted to the adjacent particle following it.
5. We need to keep creating disturbance in order to propagate wave (energy or disturbance) continuously.

(a) Waves classification

The waves are classified under two high level headings :

1. Mechanical waves : The motion of the particle constituting the medium follows mechanical laws i.e. Newton's laws of motion. Mechanical waves originate from a distrubance in the medium (such as a stone dropping in a pond) and the disturbance propagates through the medium. The force between the atoms in the medium are responsible for the propagation of mechanical waves. Each atom exerts a force on the atoms near it, and through this force the motion of the atom is transmitted to the others. The atoms in the medium do not experience any net displacement.
Mechanical waves is further classified in two categories such that
1. Transverse waves (waves on a string)
2. Longitudnal waves (sound waves)

2. Non Mechanical waves : These are electro magnetic waves. The electromagnetic waves do not require a medium for propagation. Its speed in vacuum is a universal constant. The motion of the electromagnetic waves in a medium depends on the electromagnetic properties of the medium.

Transverse waves

If the disturbance travels in the x direction but the particles move in a direction, perpendicular to the x axis as the wave passes it is called a transverse waves.

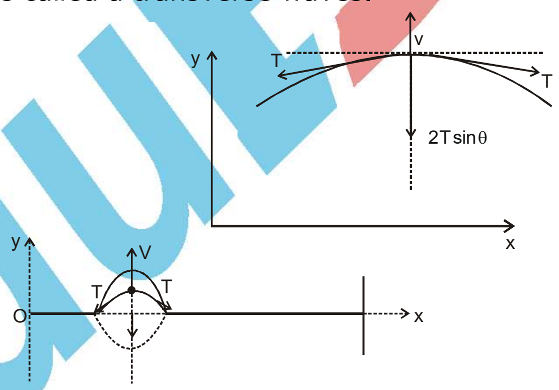


figure - I

Consider a sinusoidal harmonic wave travelling through a string and the motion of a particle as shown in the figure Ist (only one unit of wave shown for illustration purpose). Since the particle is displaced from its natural (mean) position, the tension in the string arising from the deformation tends to restore the position of the particle. On the other hand, velocity of the particle (kinetic energy) move the particle farther is zero. Therefore, the particle is pulled down due to tension towards mean position. In the process, it acquires kinetic energy (greater speed) and overshoots the mean position in the downward direction. The cycle of restoration of position continues as vibration (oscillation) of particle takes place.

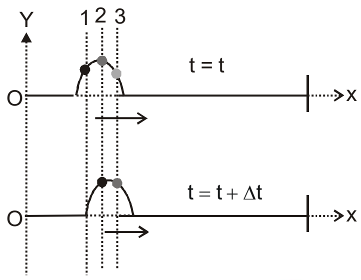
Longitudinal waves

Longitudinal waves are characterized by the direction of vibration (disturbance) and wave motion. They are along the same direction. It is clear that vibration in the same direction needs to be associated with a "restoring" mechanism in the longitudinal direction.

(b) Mathematical description of waves

We shall attempt here to evolve a mathematical model of a travelling transverse wave. For this, we choose a specific set up of string and associated transverse wave travelling through it. The string is tied to a fixed end, while disturbance is imparted at the free end by up and down motion. For our purpose, we consider that pulse is small in dimension; the string is light, elastic and homogeneous. The assumptions are required as we visualize a small travelling pulse which remains undiminished when it moves through the strings. We also assume that the string is long enough so that our observation is not subjected to pulse reflected at the fixed end.

For understanding purpose, we first consider a single pulse as shown in the figure (irrespective of whether we can realize such pulse in practice or not). Our objective here is to determine the nature of a mathematical description which will enable us to determine displacement (disturbance) of string as pulse passes through it. We visualize two snap shots of the travelling pulse at two close time instants "t" and "t + Δt". The single pulse is moving towards right in the positive x-direction.



The vibration and wave motion are at right angle to each other.

Three position along x-axis named "1", "2" and "3" are marked with three vertical dotted lines. At either of two instants as shown, the positions of string particles have different displacements from the undisturbed position on horizontal x-axis. We can conclude from this observation that displacement in y-direction is a function of positions of particle in x-direction. As such, the displacement of a particle constituting the string is a function of "x".

Let us now observe the positions of a given particle, say "1". It has certain positive displacement at time t = t, At the next snapshot at t = t + Δt, the displacement has reduced to zero. The particle at "2" has maximum displacement at t = t, but the same has reduced at t = t + Δt. The third particle at "3' has certain positive displacement at t = t, At t = t + Δt, it acquires additional positive displacement and reaches the position of maximum displacement. From these observation, we conclude that displacement of a particle at any position along the string is a function of "t".

Combining two observations, we conclude that displacment of a particle is a function of both position of the particle along the string and time.

$$y = f(x, t)$$

We can further specify the nature of the mathematical function by association the speed of the wave in our consideration. Let "v" be the constant speed with which wave travels from the left end to the right end. We notice that wave function at a given position of the string is a function of time only as we are considering displacement at a particular value of "x". Let us consider left hand end of the string as the origin of reference (x = 0 and t = 0). The displacement in y-direction (disturbance) at x = 0 is a function of time, "t" only :

$$y = f(t) = A \sin \omega t$$

The disturbance travels to the right at constant speed "v'. Let it reaches a point specified as x = x after time "t". If we visualize to describe the origin of this disturbance at x = 0, then time elapsed for the distrubance to move from the origin (x = 0) to the point (x = x) is "x/v". Therefore, if we want to use the function of displacement at x = 0 as given above, then we need to subtract the time elapsed and set the equation is :

$$y = f\left(t - \frac{x}{v}\right) = A \sin\omega\left(t - \frac{x}{v}\right)$$

This can also be expressed as

$$\Rightarrow \quad f\left(\frac{vt - x}{v}\right) \quad \Rightarrow \quad -f\left(\frac{x - vt}{v}\right)$$

$$y(x, t) = g(x - vt)$$

using any fixed value of t (i.e. at any instant), this shows shape of the string.

If the wave is travelling in –x direction, the wave equation is written as

$$y(x, t) = f\left(t + \frac{x}{v}\right)$$

The quantity x − vt is called phase of the wave function. As phase of the pulse has fixed value

$x - vt = \text{const.}$

Taking the derivative w.r.t. time $\frac{dx}{dt} = v$

where v is the phase velocity although often called wave velocity. It is the velocity at which a particular phase of the distrubance travels through space.

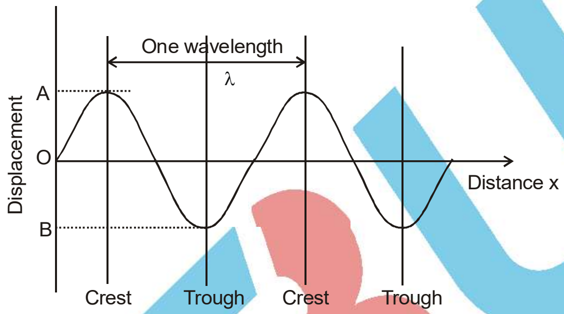
In order for the function to represent a wave travelling at speed v, the quantities x, v and t must appear in the combination (x + vt) or (x − vt). Thus $(x - vt)^2$ is acceptable but $x^2 - v^2t^2$ is not.

(c) Describing Waves :

Two kinds of graph may be drawn displacement - distance and displacement-time.

A displacement-distance graph for a transverse mechanical waves shows the displacement y of the vibrating particles of the transmitting medium at different distance x from the source at a certain instant i.e. it is like a photograph showing shape of the wave at that particular instant.

The maximum displacement of each particle from its undisturbed position is the amplutude of the wave. In the figure 1, it OA or OB.



The wavelength λ of a wave is generally taken as the distance between two successive crests or two successive trough. To be more specific, it is the distance between two consecutive points on the wave which have same phase.

A displacement-time graph may also be drawn for a wave motion, showing how the displacement of one particle at a particular distance from the source varies with time. If this is simple harmonic variation then the graph is a sine curve.

- **Wave Length, Frequency, Speed**

If the source of a wave makes f vibrations per second, so they will the particles of the transmitting medium. That is, the frequency of the waves equals frequency of the source.

When the source makes one complete vibration, one wave is generated and the disturbance spreads out a distance λ from the source. If the source continues to vibrate with constant frequency f, then f waves will be produced per second and the wave advances a distance f λ in one second. If v is the wave speed then

$v = f\lambda$

This relationship holds for all wave motions.

☛ Frequency depends on source (not on medium), v depends on medium (not on source frequency), but wavelength depend on both medium and source.

(d) Initial Phase :

At x = 0 and t = 0, the sine function evaluates to zero and as such y-displacement is zero. However, a wave form can be such that y-displacement is not zero at x =0 and t = 0. In such case, we need to account for the displacement by introducting an angle like :

$y(x,t) = A\sin(kx - \omega t + \phi)$

where "ϕ" is initial phase. At x = 0 and t = 0.

$y(0, 0) = A\sin(\phi)$

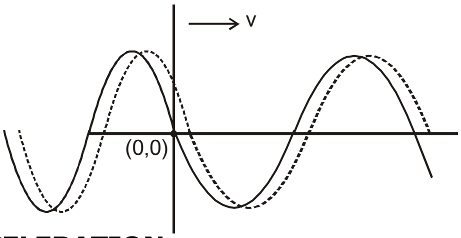
The measurement of angle determines following two aspects of wave form at x = 0, t = 0 : (i) whether the displacement is positive or negative and (ii) whether wave form has positive or negative slope.

For a harmonic wave represented by sine function, there are two values of initial phase angle for which displacement at reference origin (x = 0, t = 0) is positive and has equal magnitude. We know that the sine values of angles in first and second quadrants are positive. A pair of initial phase angles, say $\phi = \pi/3$ and $2\pi/3$, correspond to equal positive sine values are :

$\sin\theta = \sin(\pi - \theta)$

$$\sin\frac{\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}$$

To choose the initial phase in between the two values $\pi/3$ & $\frac{2\pi}{3}$. We can look at a wave motion in yet another way. A wave form at an instant is displaced by a distance Δx in very small time interval Δt then then speed to the particle at t = 0 & x = 0 is in upward +ve direction in further time Δt



2. PARTICLE VELOCITY AND ACCELERATION :

Particle velocity at a given position x = x is obtained by differentiating wave function with respect to time "t". We need to differentiate equation by treating "x" as constant. The partial differentiation yields particle velocity as :

$$v_p = \frac{\partial}{\partial t} y(x,t) = \frac{\partial}{\partial t} A\sin(kx - \omega t) = -\omega A \cos(kx - \omega t)$$

We can use the property of cosine function to find the maximum velocity. We obtain maximum speed when cosine function evaluates to "–1" :

$$\Rightarrow v_{pmax} = \omega A$$

The acceleration of the particle is obtained by differentiating expression of velocity partially with respect to time :

$$\Rightarrow a_p = \frac{\partial}{\partial t} v_p = \frac{\partial}{\partial t}\{-\omega A\cos(kx - \omega t)\} = -\omega^2 A \sin(kx - \omega t) = -\omega^2 y$$

Again the maximum value of the acceleration can be obtained using property of sine function :

$$\Rightarrow a_{pmax} = \omega^2 A$$

3. DIFFERENT FORMS OF WAVE FUNCTION :

Different forms give rise to bit of confusion about the form of wave function. The forms used for describing wave are :

$$y(x, t) = A \sin(kx - \omega t)$$

$$y(x, t) = A \sin(\omega t - kx + \pi)$$

Which of the two forms is correct ? In fact, both are correct so long we are in a position to accurately interpret the equation. Starting with the first equation and using trigonometric identity :

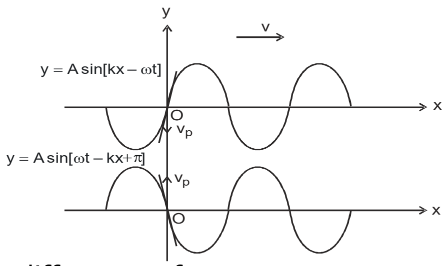
We have,

$$\Rightarrow A \sin(kx - \omega t) = A \sin(\pi - kx + \omega t) = A \sin(\omega t - kx + \pi)$$

Thus we see that two forms represent waves along at the same speed $\left(v = \frac{\omega}{k}\right)$. They differ, however, in phase. There is phase difference of "π". This has implication on the waveform and the manner particle oscillates at any given time instant and position. Let us consider two waveforms at x = 0, t = 0. The slopes of the waveforms are :

$$\frac{\partial}{\partial x} y(x,t) = kA\cos(kx - \omega t) = kA = \text{a positive number}$$

and $$\frac{\partial}{\partial x} y(x,t) = -kA\cos(\omega t - kx) = -kA = \text{a negative number}$$



Forms of wave functions

Exchange of terms in the argument of sine function results in a phase difference of π.

In the first case, the slope is positive and hence particle velocity is negative. It means particle is moving from reference origin or mean position to negative extreme position. In the second case, the slope is negative and hence particle velocity is positive. It means particle is moving from positive extreme position to reference origin or mean position. Thus two forms represent waves which differ in direction in which particle is moving at a given position.

Once we select the appropriate wave form, we can write wave equation in other forms as given here :

$$y(x, t) = A \sin (kx - \omega t) = A \sin k\left(x - \frac{\omega t}{k}\right) = A \sin \frac{2\pi}{\lambda}(x - vt)$$

Further, substituting for "k" and "ω" in wave equation, we have :

$$y(x, t) = A \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) = A \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

If we want to represent waveform moving in negative "x" direction, then we need to replace "t" by "–t".

4. THE LINEAR WAVE EQUATION :

By using wave function $y = A \sin (\omega t - kx + \phi)$, we can describe the motion of any point on the string. Any point on the string moves only vertically, and so its x coordinate remains constant. The transverse velocity v_y of the point and its transverse acceleration a_y are therefore.

$$v_y = \left[\frac{dy}{dt}\right]_{x=\text{constant}} \Rightarrow \frac{\partial y}{\partial t} = \omega A \cos (\omega t - kx + \phi) \quad ...(1)$$

$$a_y = \left[\frac{dv_y}{dt}\right]_{x=\text{constant}} \Rightarrow \frac{\partial v_y}{\partial t} = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin (\omega t - kx + \phi) \quad ..(2)$$

and hence

$v_{y.\ max} = \omega A$

$a_{y.max} = \omega^2 A$

The transverse velocity and transverse acceleration of any point on the string do not reach their maximum value simultaneously. Infact, the transverse velocity reaches its maximum value (ωA) when the displacement $y = 0$, whereas the transverse acceleration reaches its maximum magnitudes ($\omega^2 A$) when $y = \pm A$

further

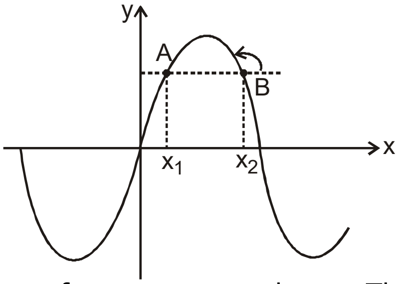
$$\left[\frac{dy}{dx}\right]_{t=\text{constant}} \Rightarrow \frac{\partial y}{\partial x} = -kA \cos (wt - kx + \phi) \quad ...(3)$$

$$= \frac{\partial^2 y}{\partial x^2} = -k^2 A \sin (\omega t - kx + \phi) \quad ...(4)$$

From (1) and (3) $\frac{\partial y}{\partial t} = -\frac{\omega}{k}\frac{\partial y}{\partial x}$

$\Rightarrow \quad v_p = -v_w \times \text{slope}$

i.e. if the slope at any point is negative, particle velocity and vice-versa, for a wave moving along positive x axis i.e. v_w is positive.



For example, consider two points A and B on the y-curve for a wave, as shown. The wave is moving along positive x-axis.

Slope at A is positive therefore at the given moment, its velocity is negative. That means it is coming downward. Reverse is the situation for particle at point B.

Now using equation (2) and (4)

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

This is known as the linear wave equation or diffential equation representation of the travelling wave model. We have developed the linear wave equation from a sinusoidal mechanical wave travelling through a medium. But it is much more general. The linear wave equation successfully describes waves on strings, sound waves and also electromagnetic waves.

Thus, the above equation can be written as,

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(i)$$

The general solution of this equation is of the form

$$y(x, t) = f(ax \pm bt) \qquad ...(ii)$$

Thus, any function of x and t which satisfies Eq. (i) or which can be written as Eq. (ii) represents a wave. The only condition is that it should be finite everywhere and at all times. Further, if these conditions are satisfied, then speed of wave (v) is given by,

$$v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{b}{a}$$

Thus plus (+) sign between ax and bt implies that the wave is travelling along negative x-direction and minus (–) sign shows that it is travelling along positive x-direction.

5. SPEED OF A TRANSVERSE WAVE ON A STRING

Consider a pulse travelling along a string with a speed v to the right. If the amplitude of the pulse is small compared to the length of the string, the tension T will be approximately constant along the string. In the reference frame moving with speed v to the right, the pulse in stationary and the string moves with a speed v to the left. Figure shows a small segment of the string of length Δl. This segment forms part of a circular arc of radius R. Instantaneously the segment is moving with speed v in a circular path, so it has centripetal acceleration v^2/R. The forces acting on the segment are the tension T at each end. The horizontal component of these forces are equal and opposite and thus cancel. The vertical component of these forces point radially inward towards the centre of the circular. arc. These radial forces provide centripetal acceleration. Let the angle substended by the segment at centre be 2θ. The net radial force acting on the segment is

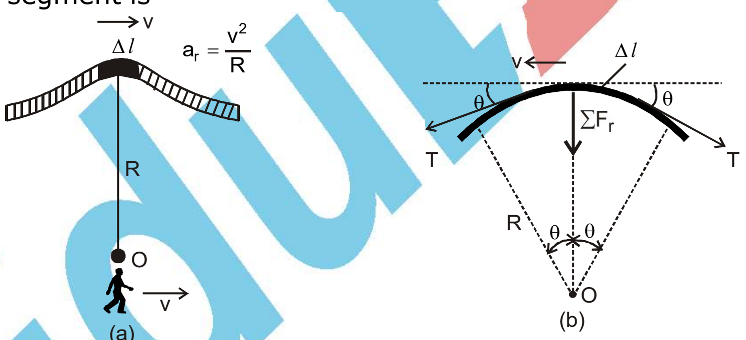


Fig. (a) To obtain the speed v of a wave on a stretched string. It is convenient to describe the motion of a small segment of the string in a moving frame of reference.

Fig. (b) In the moving frame of reference, the small segment of length Δl moves to the left with speed v. The net force on the segment is in the radial direction because the horizontal components of the tension force cancel.

$$\sum F_r = 2T\sin\theta = 2T\theta$$

Where we have used the approximation $\sin\theta \approx \theta$ for small θ.

If μ is the mass per unit length of the string, the mass of the segment of length Δl is

$$m = \mu\Delta l = 2\mu R\theta \qquad (\text{as } \Delta l = 2R\theta)$$

From Newton's second law $\quad \sum F_r = ma = \frac{mv^2}{R}$

or $\quad 2T\theta = (2\mu R\theta)\left(\frac{v^2}{R}\right) \quad \therefore \quad v = \sqrt{\frac{T}{\mu}}$

6. ENERGY CALCULATION IN WAVES :

(a) Kinetic energy per unit length

The velocity of string element in transverse direction is greatest at mean position and zero at the extreme positions of waveform. We can find expression of transverse velocity by differentiating displacement with respect to time. Now, the y-displacement is given by :

$$y = A \sin (kx - \omega t)$$

Differentiating partially with respect to time, the expression of particle velocity is :

$$v_p = \frac{\partial y}{\partial t} = -\omega A \cos (kx - \omega t)$$

In order to calculate kinetic energy, we consider a small string element of length "dx" having mass per unit length "μ". The kinetic energy of the element is given by :

$$dK = \frac{1}{2} dm v_p^2 = \frac{1}{2} \mu dx \omega^2 A^2 \cos^2 (kx - \omega t)$$

This is the kinetic energy associated with the element in motion. Since it involves squared of cosine function, its value is greatest for a phase of zero (mean position) and zero for a phase of $\frac{\pi}{2}$ (maximum displacement).

Now, we get kinetic energy per unit length, "K_L", by dividing this expression with the length of small string considered :

$$K_L = \frac{dK}{dx} = \frac{1}{2} \mu \omega^2 A^2 \cos^2 (kx - \omega t)$$

- **Rate of transmission of kinetic energy**

The rate, at which kinetic energy is transmitted, is obtained by dividing expression of kinetic energy by small time element, "dt" :

$$\frac{dK}{dt} = \frac{1}{2} \mu \frac{dx}{dt} \omega^2 A^2 \cos^2 (kx - \omega t)$$

But, wave or phase speed, v, is time rate of position i.e. $\frac{dx}{dt}$. Hence,

$$\frac{dK}{dt} = \frac{1}{2} \mu v \omega^2 A^2 \cos^2 (kx - \omega t)$$

Here kinetic energy is a periodic function. We can obtain average rate of transmission of kinetic energy by integrating the expression for integral wavelengths. Since only $\cos^2(kx - \omega t)$ is the varying entity, we need to find average of this quantity only. Its integration over intergal wavelengths give a value of "$\frac{1}{2}$".

Hence, average rate of transmission of kinetic energy is :

$$\left.\frac{dK}{dt}\right|_{avg} = \frac{1}{2} \times \frac{1}{2} \mu v \omega^2 A^2 = \frac{1}{4} \mu v \omega^2 A^2$$

(b) Elastic potential energy

The elastic potential energy of the string element results as string element is stretched during its oscillation. The extension or stretching is maximum at mean position. We can see in the figure that the length of string element of equal x-length "dx" is greater at mean position than at the extreme. As a matter of fact, the elongation depends on the slope of the curve. Greater the slope, greater is the elongation. The string has the least length when slope is zero. For illustration purpose, the curve is purposely drawn in such a manner that the elongation of string element at mean position is highlighted.

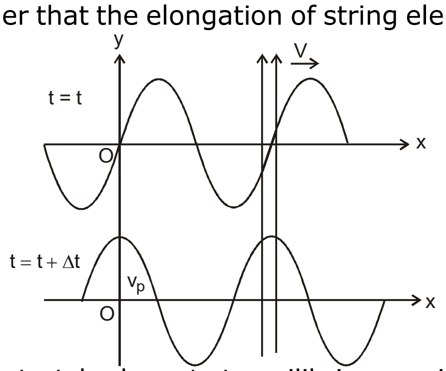


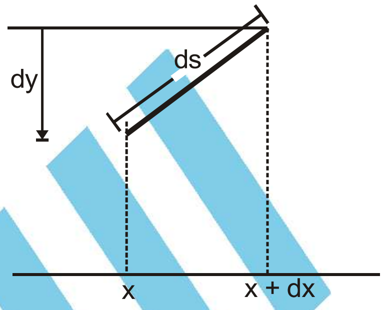
fig : The string element stretched most at equilibrium position

Greater extension of string element corresponds to greater elastic energy. As such, it is greatest at mean position and zero at extreme position. This deduction in contrary to the case of SHM in which potential energy is greatest at extreme position and zero at mean position.

- **Potential energy per unit length**

When the string segment is stretched from the length dx to the length ds an amount of work = T (ds – dx) is done. This is equal to the potential energy stored in the stretched string segment. So the potential energy in this case is :

$$U = T\,(ds - dx)$$



Now $\quad ds = \sqrt{(dx^2 + dy^2)}$

$$= dx\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

from the binomial expansion

so $\quad ds \approx dx + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$

$$U = T\,(ds - dx) \approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

or the potential energy density

$$\frac{dU}{dx} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \qquad \text{...(i)}$$

$$\frac{dy}{dx} = kA\cos(kx - \omega t)$$

and $T = v^2 \mu$

Put above value in equation (i) then we get

$$\frac{dU}{dx} = \frac{1}{2}\mu\omega^2A^2\cos^2(kx - \omega t)$$

- **Rate of transmission of elastic potential energy**

The rate, at which elastic potential energy is transmitted, is obtained by dividing expression of kinetic energy by small time element, "dt". This expression is same as that for kinetic enegy.

$$\frac{dU}{dt} = \frac{1}{2}\mu v\omega^2A^2\cos^2(kx - \omega t)$$

and average rate of transmission of elastic potential energy is :

$$\frac{dU}{dt}\Big|_{avg} = \frac{1}{2} \times \frac{1}{2}\mu v\omega^2A^2 = \frac{1}{4}\mu v\omega^2A^2$$

(c) Mechanical energy per unit length

Since the expression elastic potential energy is same as that of kinetic energy, we get mechanical energy expression by multiplying expression of kinetic energy by "2". The mechanical energy associated with small string element, "dx", is :

$$dE = 2x dK = 2x\frac{1}{2}dmv_p^2 = \mu dx\omega^2A^2\cos^2(kx - \omega t)$$

Similarly, the mechanical energy per unit length is :

$$E_L = \frac{dE}{dx} = 2x\frac{1}{2}\mu\omega^2A^2\cos^2(kx - \omega t) = \mu\omega^2A^2\cos^2(kx - \omega t)$$

(d) Average power transmitted

The average power transmitted by wave is equal to time rate of transmission of mechanical energy over integral wavelengths. It is equal to :

$$P_{avg} = \frac{dE}{dt}\Big|_{avg} = 2 \times \frac{1}{4}\mu v\omega^2A^2 = \frac{1}{2}\mu v\omega^2A^2$$

If mass of the string is given in terms of mass per unit volume, "ρ", then we make appropriate change in the derivation. We exchange "μ" by "ρs" where "s" is the cross section of the string :

$$P_{avg} = \frac{1}{2}\rho s v \omega^2 A^2$$

(e) Energy density

Since there is no loss of energy involved, it is expected that energy per unit length is uniform throughout the string. As much energy enters that much energy goes out for a given length of string. This average value along unit length of the string length is equal to the average rate at which energy is being transferred.

The average mechanical energy per unit length is equal to integration of expression over integral wavelength

$$E_L|_{avg} = 2x\frac{1}{4}\ \mu v \omega^2 A^2 = \frac{1}{2}\mu v \omega^2 A^2$$

We have derived this expression for harmonic wave along a string. The concept, however, can be extended to two or three dimensional transverse waves. In the case of three dimensional transverse waves, we consider small volumetric element. We, then, use density, ρ, in place of mass per unit length, μ. The corresponding average energy per unit volume is referred as energy density (u) :

$$u = \frac{1}{2}\rho v w^2 A^2$$

(f) Intensity

Intensity of wave (I) is defined as power transmitted per unit cross section area of the medium :

$$I = \rho s v \omega^2 \frac{A^2}{2s} = \frac{1}{2}\rho v w^2 A^2$$

Intensity of wave (I) is a very useful concept for three dimensional waves radiating in all direction from the source. This quantity is usually referred in the context of light waves, which is transverse harmonic wave in three dimensions. Intensity is defined as the power transmitted per unit cross sectional area. Since light spreads uniformly all around, intensity is equal to power transmitted, divided by spherical surface drawn at that point with source at its center.

☛ **Phase difference between two particles in the same wave :**

The general expression for a sinusoidal wave travelling in the positive x direction is

$y(x, t) = A \sin(\omega t - kx)$

Eq^n of Particle at x_1 is given by $y_1 = A \sin(\omega t - kx_1)$

Eq^n of particle which is at x_2 from the origin

$y_2 = A\sin(\omega t - kx_2)$

Phase difference between particles is $k(x_2 - x_1) = \Delta\phi$

$$K\Delta x = \Delta\phi \Rightarrow \quad \Delta x \Rightarrow \frac{\Delta\phi}{k}$$

7. PRINCIPLE OF SUPERPOSITION :

This principle defines the displacement of a medium particle when it is oscillating under the influence of two or more than two waves. The principle of superposition is stated as :

"When two or more waves superpose on a medium particle than the resultant displacement of that medium particle is given by the vector sum of the individual displacements produced by the component waves at that medium particle independently."

Let $\vec{y}_1, \vec{y}_2, \ldots\ldots \vec{y}_N$ are the displacements produced by N independent waves at a medium particle in absence of others then the displacemnt of that medium, when all the waves are superposed at that point, is given as

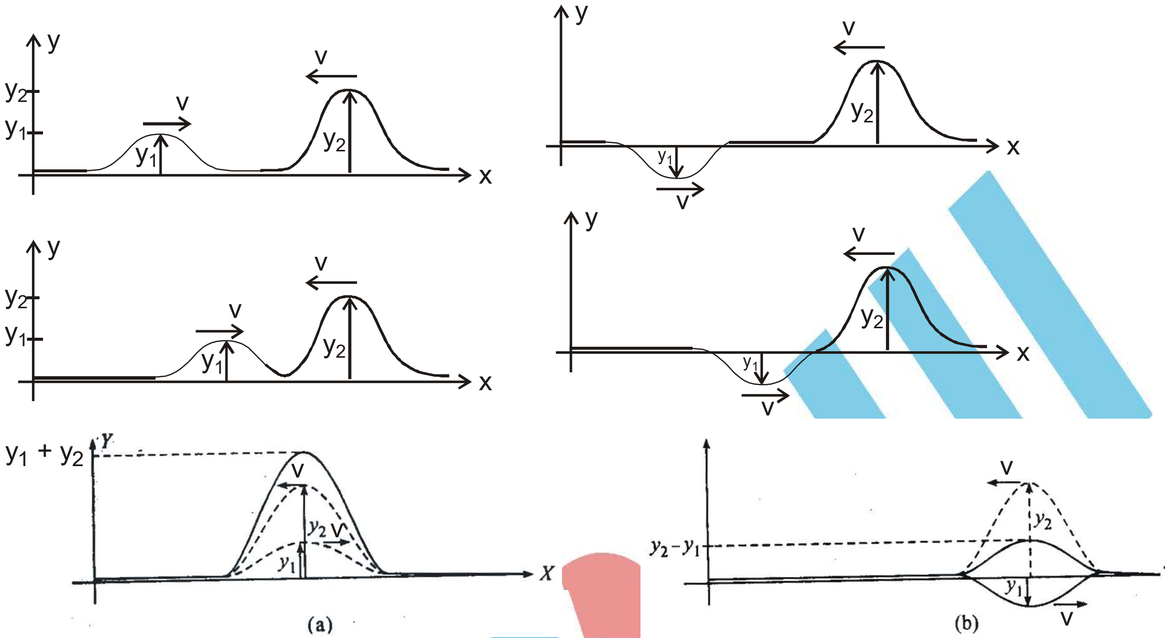
$$\vec{y} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \ldots\ldots + \vec{y}_N$$

If all the waves are producing oscillations at that point are collinear then the displacement of the medium particle where superposition is taking place can be simply given by the algebric sum of the individual displacement. Thus we have

$$y = y_1 + y_2 + \ldots\ldots\ldots\ldots + y_N$$

The above equation is valid only if all individual displacements $y_1, y_2 \ldots\ldots\ldots\, y_N$ are along same straight line.

A simple example of superposition can be understood by figure shown. Suppose two wave pulses are travelling simultaneously in opposite directions as shown. When they overlap each other the displacement of particle on string is the algebric sum of the two displacement as the displacements of the two pulses are in same direction. Figure shown (b) also shows the similar situation when the wave pulses are in opposite side.



(a) Applications of Principle of Superposition of Waves

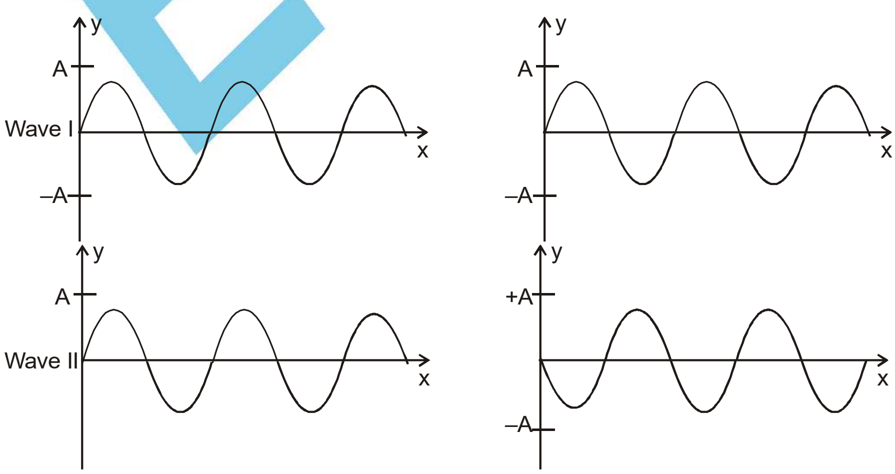
There are several different phenomenon which takes place during superposition of two or more wave depending on the wave characteristics which are being superposed. We'll discuss some standard phenomenons, and these are :

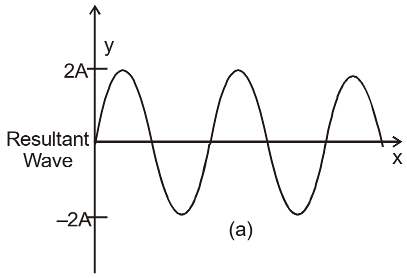
(1) Interference of Wave
(2) Stationary Waves
(3) Beats
(4) Lissajou's Figures (Not discussed here in detail.)

Lets discuss these in detail.

(b) Interference of Waves

Suppose two sinusoidal waves of same wavelength and amplitude travel in same direction along the same straight line (may be on a stretched string) then superposition principle can be used to define the resultant displacement of every medium particle. The resultant wave in the medium depends on the extent to which the waves are in phase with respect to each other, that is, how much one wave form is shifted from the other waveform. If the two waves are exactly in same phase, that is the shape of one wave exactly fits on to the other wave then they combine to double the displacement of every medium particle as shown in figure (a). This phenomenon we call as constructive interference. If the superposing waves are exactly out of phase or in opposite phase then they combine to cancel all the displacements at every medium particle and medium remains in the form of a straight line as shown in figure (b)





(a)

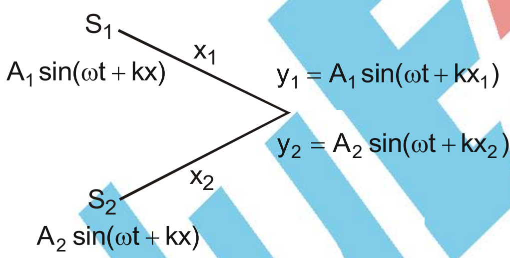
(b)

This phenomenon we call destructive interference. Thus we can state that when waves meet, they interfere constructively if they meet in same phase and destructively if they meet in opposite phase. In either case the wave patterns do not shift relative to each other as they propagates. Such superposing waves which have same form and wavelength and have a fixed phase relation to each other, are called coherent waves. Sources of coherent waves are called coherent source. Two indepedent sources can never be coherent in nature due to practical limitations of manufacturing process. Generally all coherent sources are made either by spliting of the wave forms of a single source or the different sources are fed by a single main energy source.

In simple words interference is the phenomenon of superposition of two coherent waves travelling in same direction.

We've discussed that the resultant displacement of a medium particle when two coherent waves interfere at that point, as sum or difference of the individual displacements by the two waves if they are in same phase (phase difference = 0, 2π, ...) or opposite phase (phase difference = π, 3π,.....) respectively. But the two waves can also meet at a medium particle with phase difference other then 0 or 2π, say if phase difference ϕ is such that $0 < \phi < 2\pi$, then how is the displacement of the point of superposition given ? Now we discuss the interference of waves in details analytically.

(c) Analytical Treatment of Interference of Waves



Interference implies super position of waves. Whenever two or more than two waves superimpose each other they give sum of their individual diplacement.

Let the two waves coming from sources S_1 & S_2 be

$y_1 = A_1 \sin(\omega t + kx_1)$

$y_2 = A_2 \sin(\omega t + kx_2)$ respectively.

Due to superposition

$y_{net} = y_1 + y_2$

$y_{net} = A_1 \sin(\omega t + kx_1) + A_2 \sin(\omega t + kx_2)$

Phase difference between y_1 & $y_2 = k(x_2 - x_1)$

i.e., $\Delta\phi = k(x_2 - x_1)$

As $\Delta\phi = \dfrac{2\pi}{\lambda}\Delta x$ (where Δx = path difference & $\Delta\phi$ = phase difference)

$$A_{net} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

$$\Rightarrow \quad A_{net}^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\phi$$

$$\therefore \quad I_{net} = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \quad (\text{as } I \propto A^2)$$

When the two displacements are in phase, then the resultant amplitude will be sum of the two amplitude & I_{net} will be maximum, this is known of constructive interference.

For I_{net} to be maximum

$\cos\phi = 1 \Rightarrow \phi = 2n\pi$ where n = {0,1,2,3,4,5...........}

$$\frac{2\pi}{\lambda}\Delta x = 2n\pi \Rightarrow \mathbf{\Delta x = n\lambda}$$

For constructive interference

$$I_{net} = (\sqrt{I_1} + \sqrt{I_2})^2$$

When $I_1 = I_2 = I$

$I_{net} = 4\,I$

$A_{net} = A_1 + A_2$

When superposing waves are in opposite phase, the resultant amplitude is the difference of two amplitudes & I_{net} is minimum; this is known as destructive interference.

For I_{net} to be minimum,

$\cos\Delta\phi = -1$

$\Delta\phi = (2n + 1)\pi$ where n = {0,1,2,3,4,5..........}

$$\frac{2\pi}{\lambda}\Delta x = (2n+1)\pi \Rightarrow \mathbf{\Delta x = (2n+1)\frac{\lambda}{2}}$$

For destructive interfence

$$I_{net} = (\sqrt{I_1} - \sqrt{I_2})^2$$

If $I_1 = I_2$

$I_{net} = 0$

$A_{net} = A_1 - A_2$

Ratio of I_{max} & $I_{min} = \dfrac{(\sqrt{I_1} + \sqrt{I_2})^2}{(\sqrt{I_1} - \sqrt{I_2})^2}$

Generally,

$I_{net} = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$

If $I_1 = I_2 = I$

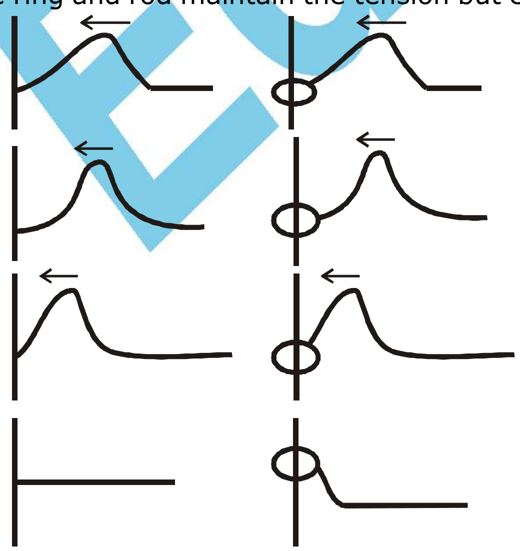
$I_{net} = 2I + 2I\cos\phi$

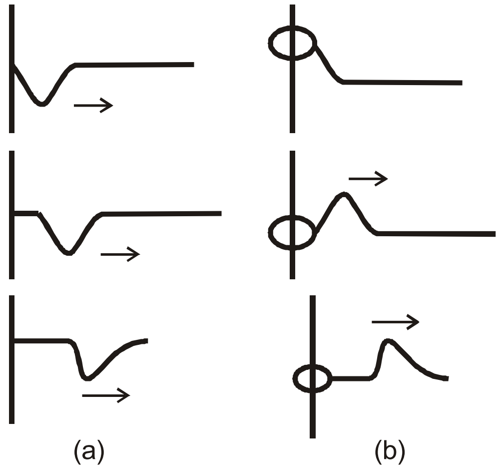
$I_{net} = 2I(1 + \cos\phi) = 4I\cos^2\dfrac{\Delta\phi}{2}$

8. REFLECTION AND TRANSMISSION IN WAVES :

1. When a pulse travelling along a string reaches the end, it is reflected. If the end is fixed as shown in figure (a), the pulse returns inverted. This is bacause as the leading edge reaches the wall, the string pulls up the wall. According to Newton's third law, the wall will exert an equal and opposite force on the string as all instants. This force is therefore, directed first down and then up. It produces a pulse that is inverted but otherwise identical to the original.

The motion of free end can be studied by letting a ring at the end of string sliding smoothly on the rod. The ring and rod maintain the tension but exert no transverse force.





Reflection of wave pulse (a) at a fixed end of a string and (b) at a free end. Time increases from top to bottom in each figure.

When a wave arrives at this free end, the ring slides the rod. The ring reaches a maximum displacement. At this position the ring and string come momentarily to rest as in the fourth drawing from the top in figure (b). But the string is stretched in this position, giving increased tension, so the free end of the string is pulled back down, and again a reflected pulse is produced, but now the direction of the displacement is the same as for the initial pulse.

2. The formation of the reflected pulse is similar to the overlap of two pulses travelling in opposite directions. The net displacement at any point is given by the principle of superposition.

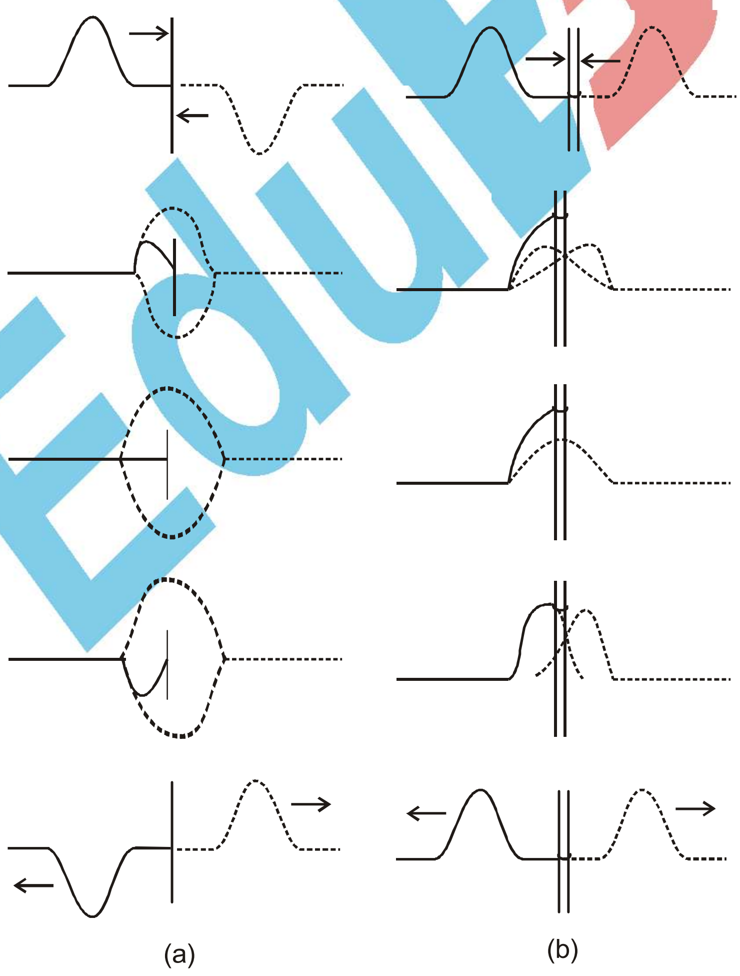


Fig (a) : shows two pulses with the same shape, one inverted with respect to the other, travelling in opposite directions. Because these two pulses have the same shape the net displacement of the point where the string is attached to the wall is zero at all times.

Fig (b) : shows two pulses with the same shape, travelling in oppoiste directions but not inverted relative to each other. Note that at one instant, the displacement of the free end is double the pulse height.

9. REFLECTION AND TRANSMISSION BETWEEN TWO STRING :

Here we are dealing with the case where the end point is neither completely fixed nor completely free to move As we consider an example where a light string is attached to a heavy string as shown is figure a. If a wave pulse is produced on a light string moving towards the friction a part of the wave is reflected and a part is transmitted on the heavier string the reflected wave is inverted with respect to the original one.

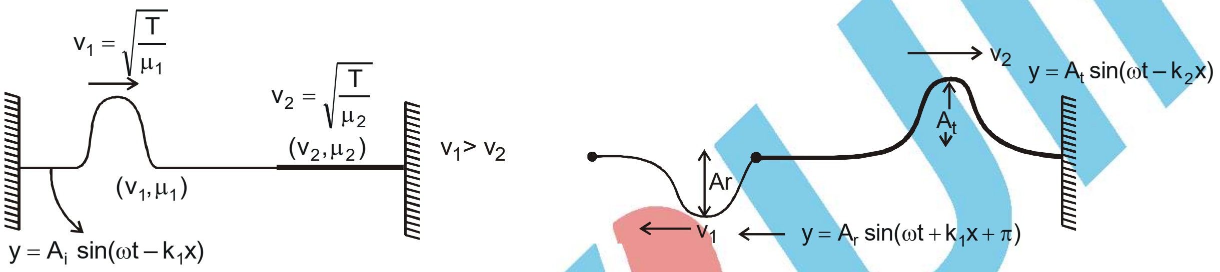


figure (a)

On the other hand if the wave is produced on the heavier string which moves toward the junction a part will the reflected and a part transmitted, no inversion in waves shape will take place.

The wave velocity is smaller for the heavier string lighter string

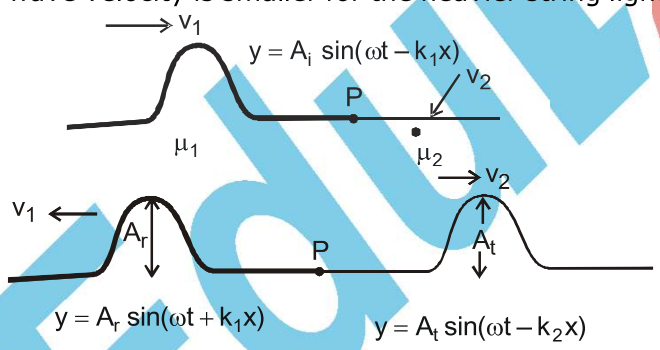


figure : (b)

Now to find the relation between A_i, A_r, A_t we consider the figure (b)

Incident Power = Reflected Power + Transmitted Power

$$P_i = P_r + P_t$$

$$2\pi^2 f^2 A_i^2 \mu_1 v_1 = 2\pi^2 f^2 A_r^2 \mu_1 v_1 + 2\pi^2 f^2 A_t^2 \mu_2 v_2 \quad \ldots(i)$$

Put $\mu_1 = \frac{T}{v_1^2}$ and $\mu_2 = \frac{T}{v_2^2}$

in equation (i) their

$$\frac{A_i^2}{v_1} = \frac{A_r^2}{v_1} + \frac{A_t^2}{v_2}$$

$$A_i^2 - A_r^2 = \frac{v_1}{v_2}A_t^2 \qquad ....(ii)$$

Maximum displacement of joint particle P (as shown in figure) due to left string

$= A_i + A_r$

Maximum displacement of joint particle due to right string $= A_t$

At the boundary (at point P) the wave must be continuous, that is there are no kinks in it. Then we must have $A_i + A_r = A_t$ ...(iii)

from equation (ii) & (iii)

$$A_i - A_r = \frac{v_1}{v_2}A_t \qquad ...(iv)$$

from eq. (iii) & (iv)

$$A_t = \left[\frac{2v_2}{v_1 + v_2}\right]A_i$$

$$A_r = \left[\frac{v_2 - v_1}{v_1 + v_2}\right]A_i$$

10. STANDING WAVES :

In previous section we've discussed that when two coherent waves superpose on a medium particle, phenomenon of interference takes place. Similarly when two coherent waves travelling in opposite direction superpose then simultaneous interference if all the medium particles takes place. These waves interfere to produce a pattern of all the medium particles what we call, a stationary wave. If the two interfering waves which travel in opposite direction carry equal energies then no net flow of energy takes place in the region of superposition. Within this region redistribution of energy takes place between medium particles. There are some medium particles where constructive interference takes place and hence energy increases and on the other hand there are some medium particles where destructive interference takes place and energy decreases. Now we'll discuss the stationary waves analytically.

Let two waves of equal amplitude are travelling in opposite direction along x-axis. The wave equation of the two waves can be given as

$y_1 = A \sin(\omega t - kx)$ [Wave travelling in +x direction] ...(1)

and $y_2 = A \sin(\omega t + kx)$ [Wave travelling in –x direction] ...(2)

When the two waves superpose on medium particles, the resultant displacement of the medium particles can be given as

$y = y_1 + y_2$

or $y = A \sin(\omega t - kx) + A \sin(\omega t + kx)$

or $y = A [\sin\omega t \cos kx - \cos \omega t \sin kx + \sin \omega t \cos kx + \cos \omega t \sin kx]$

or $y = 2A \cos kx \sin \omega t$ ...(3)

Equation (3) can be rewritten as

$y = R \sin \omega t$ ...(4)

Where $R = 2 A \cos kx$ ...(5)

Here equation (4) is an equation of SHM. It implies that after superposition of the two waves the medium particles executes SHM with same frequency ω and amplitude R which is given by equation (5) Here we can see that the oscillation amplitude of medium particles depends on x i.e. the position of medium particles. Thus on superposition of two coherent waves travelling in opposite direction the resulting interference pattern, we call stationary waves, the oscillation amplitude of the medium particle at different positions is different.

At some point of medium the resultant amplitude is maximum which are given as

R is maximum when $\cos kx = \pm 1$

or $\frac{2\pi}{\lambda}x = N\pi$ $[N \in I]$

or $x = \frac{N\lambda}{2}$

or $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$ .....

and the maximum value of R is given as

$R_{max} = \pm 2A$ ...(6)

Thus in the medium at position $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$, ... the waves interfere constructively and the amplitude of oscillations becomes 2A. Similarly at some points of the medium, the waves interfere destructively, the oscillation amplitude become minimum i.e. zero in this case. These are the points where R is minimum, when

$\cos kx = 0$

or $\frac{2\pi x}{\lambda} = (2N+1)\frac{\pi}{2}$

or $x = (2N + 1)\frac{\lambda}{4}$ $[N \in I]$

or $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$...........

and the minimum value of R is given as

$R_{min} = 0$ [7]

Thus in the medium at position $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$ .... the waves interfere destructively and the amplitude of oscillation becomes zero. These points always remain at rest. Figure (a) shows the oscillation amplitude of different medium particles in a stationary waves.

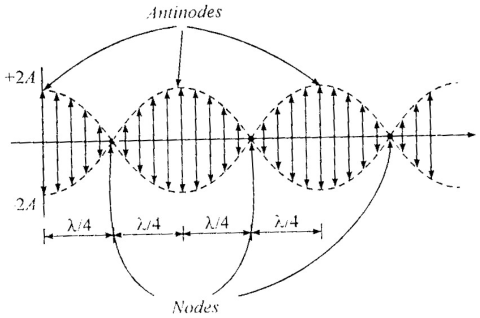


figure (a)

In figure (a) we can see that the medium particles at which constructive interference takes place are called antinodes of stationary wave and the points of destructive interference are called nodes of stationary waves which always remain at rest.

Figure (b) explain the movement of medium particles with time in the region where stationary waves are formed. Let us assume that at an instant t = 0 all the medium particles are at their extreme positions as shown in figure - (b - 1). Here points ABCD are the nodes of stationary waves where medium particles remains at rest. All other starts moving towards their mean positions and t = $T/4$ all particles cross their mean position as shown in figure (b – 3), you can see in the figure that the particles at nodes are not moving. Now the medium crosses their mean position and starts moving on other side of mean position toward the other extreme position. At time t = T/2, all the particles reach their other extreme position as shown in figure (b - 5) and at time t = 3T/4 again all these particles cross their mean position in opposite direction as shown in figure (b - 7).

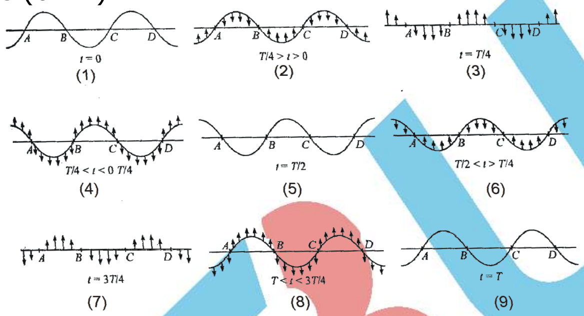


figure (b)

Based on the above analysis of one complete oscillations of the medium particles, we can make some interference for a stationary waves. These are :

(i) In oscillations of stationary wave in a region, some points are always at rest (nodes) and some oscillates with maximum amplitudes (antinodes). All other medium particles oscillate with amplitudes less then those of antinodes.

(ii) All medium particles between two successive nodes oscillate in same phase and all medium particles on one side of a node oscillate in opposite phase with those on the other side of the same node.

(iii) In the region of a stationary wave during one complete oscillation all the medium particles come in the form of a straight line twice.

(iv) If the component wave amplitudes are equal, then in the region where stationary wave is formed, no net flow of energy takes place, only redistribution of energy takes place in the medium.

(a) Different Equation for a Stationary Wave

Consider two equal amplitude waves travelling in opposite direction as

$y_1 = A \sin(\omega t - kx)$ ...(11)

and $y_2 = A \sin(\omega t + kx)$ ...(12)

The result of superposition of these two waves is

$y = 2A \cos kx \sin \omega t$ ...(13)

Which is the equation of stationary wave where 2A cos kx represents the amplitude of medium particle situated at position x and sin ωt is the time sinusoidal factor. This equation (13) can be written in several ways depending on initial phase differences in the component waves given by equation (11)) can (12). If the superposing waves are having an initial phase difference π, then the component waves can be expressed as

$y_1 = A \sin(\omega t - kx)$ ...(14)

$y_2 = -A \sin(\omega t - kx)$ ...(15)

Superposition of the above two waves will result

$y = 2A \sin kx \cos \omega t$ ...(16)

Equation (16) is also an equation of stationary wave but here amplitude of different medium particles in the region of interference is given by

$R = 2A \sin kx$ ...(17)

Similarly the possible equations of a stationary wave can be written as

$y = A_0 \sin kx \cos (\omega t + \phi)$ ....(18)

$y = A_0 \cos kx \sin (\omega t + \phi)$ ...(19)

$y = A_0 \sin kx \sin (\omega t + \phi)$ ...(20)

$y = A_0 \cos kx \cos (\omega t + \phi)$ ...(21)

Here A_0 is the amplitude of antinodes. In a pure stationary wave it is given as

$A_0 = 2A$

Where A is the amplitude of component waves. If we care fully look at equation (18) to (21), we can see that in equation (18) and (20), the particle amplitude is given by

$R = A_0 \sin kx$ ...(22)

Here at x = 0, there is nodes as R = 0 and in equation (19) and (21) the particle amplitude is given as

$R = A_0 \cos kx$ ...(23)

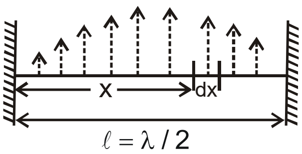
Here at x = 0, there is an antinode as $R = A_0$. Thus we can state that in a given system of co-ordinates when origin of system is at a node we use either equation (18) or (20) for analytical representation of a stationary wave and we use equation (19) or (21) for the same when an antinode is located at the origin of system.

(b) Energy of standing wave in one loop

When all the particles of one loop are at extreme position then total energy in the loop is in the form of potential energy only when the particles reaches its mean position then total potential energy converts into kinetic energy of the particles so we can say total energy of the loop remains constant

Total kinetic energy at mean position is equal to total energy of the loop because potential energy at mean position is zero.

Small kinetic energy of the particle which is in element dx is



$d(KE) = \frac{1}{2} dmv^2$

$dm = \mu\, dx$

Velocity of particle at mean position

$= 2A \sin kx\, \omega$

then $d(KE) = \frac{1}{2}\mu dx \cdot 4A^2 \omega^2 \sin^2 kx \Rightarrow d(KE) = 2A^2\omega^2 \mu \cdot \sin^2 kx\, dx$

$$\int d(K.E) = 2A^2\omega^2\mu \int_0^{\lambda/2} \sin^2 kx dx$$

$$\text{Total K.E} = A^2\omega^2\mu \int_0^{\lambda/2} (1 - \cos 2kx) dx = A^2\omega^2\mu \left[x - \frac{\sin 2kx}{2k} \right]_0^{\lambda/2} = \frac{1}{2}\lambda A^2 \omega^2 \mu$$

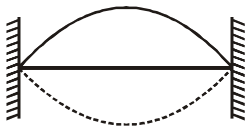
11. STATIONARY WAVES IN STRINGS :

(a) When both end of string is fixed :

A string of length L is stretched between two points. When the string is set into vibrations, a transverse progressive wave begins to travel along the string. It is reflected at the other fixed end. The incident and the reflected waves interfere to produce a stationary transverse wave in which the **ends are always nodes, if both ends of string are fixed.**

Fundamental Mode

(a) In the simplest form, the string vibrates in one loop in which the ends are the nodes and the centre is the antinode. This mode of vibration is known as the fundamental mode and frequency of vibration is known as the fundamental frequency or first harmonic.



Since the distance between consecutive nodes is $\frac{\lambda}{2}$

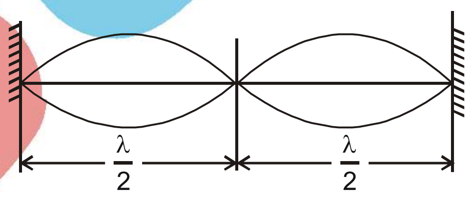
$\therefore \quad L = \frac{\lambda_1}{2} \qquad \therefore \quad \lambda_1 = 2L$

If f_1 is the fundamental frequency of vibration, then the velocity of transverse waves is given as,

$$v = \lambda_1 f_1 \qquad \text{or} \qquad f_1 = \frac{v}{2L} \qquad \text{...(i)}$$

First Overtone

(b) The same string under the same conditions may also vibrate in two loops, such that the centre is also the node



$\therefore \quad L = \frac{2\lambda_2}{2} \qquad \therefore \quad \lambda_2 = L$

If f_2 is frequency of vibrations

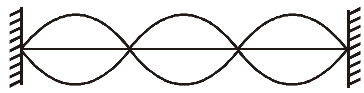
$\therefore \quad f_2 = \frac{v}{\lambda_2} = \frac{v}{L}$

$\therefore \quad f_2 = \frac{v}{L} \qquad \text{...(ii)}$

The frequency f_2 is known as second harmonic or first overtone.

Second Overtone

(c) The same string under the same conditions may also vibrate in three segments.



$\therefore \quad L = \frac{3\lambda_3}{2}$

$\therefore \quad \lambda_3 = \frac{2}{3}L$

If f_3 is the frequency in this mode of vibration, then,

$$f_3 = \frac{3v}{2L} \qquad \text{...(iii)}$$

The frequency f_3 is known as third harmonic or second overtone.

Thus a stretched string vibrates with frequencies, which are integral multiples of the fundamental frequencies. These frequencies are known as harmonics.

The velocity of transverse wave in stretched string is given as $v = \sqrt{\frac{T}{\mu}}$. Where T = tension in the string. μ = linear density or mass per unit length of string. If the string fixed at two ends, vibrates in its fundamental mode, then

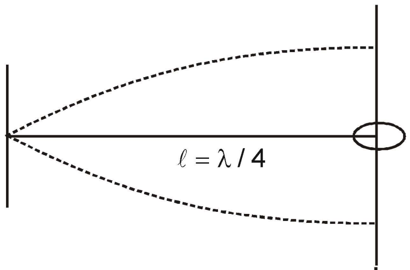
$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \qquad \text{....(17)}$$

In general $f = \frac{n}{2\ell}\sqrt{\frac{T}{\mu}}$ n^{th} harmonic

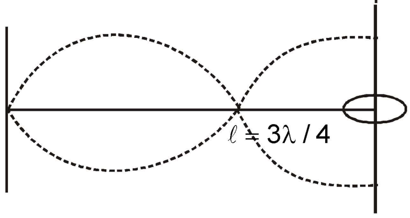
$(n - 1)^{th}$ overtone

☛ In general, any integral multiple of the fundamental frequency is an allowed frequency. These higher frequenceis are called overtones. Thus, $v_1 = 2v_0$ is the first overtone, $v_2 = 3v_0$ is the second overtone etc. An integral multiple of a frequency is called its harmonic. Thus, for a string fixed at both the ends, all the overtones are harmonics of the fundamental frequency and all the harmonics of the fundamental frequency are overtones.

(b) When one end of the string is fixed and other is free :
free end acts as antinode

1. 

$f = \frac{1}{4\ell}\sqrt{\frac{T}{\mu}}$ fundamental or I^{st} harmonic

2. 

$f = \frac{3}{4\ell}\sqrt{\frac{T}{\mu}}$ III^{rd} harmonic or I^{st} overtone

In general : $f = \frac{(2n+1)}{4\ell}\sqrt{\frac{T}{\mu}}$ $((2n + 1)^{th}$ harmonic, n^{th} overtone)

S.No.	Travelling waves	Stationary waves
1	These waves advance in a medium with a definite velocity	These waves remain stationary between two boundaries in the medium.
2	In these waves, all particles of the medium oscillate with same frequency and amplitude.	In these waves, all particles except nodes oscillate with same frequency but different amplitudes. Amplitude is zero at nodes and maximum at antinodes.
3	At any instant phase of vibration varies continuosly from one particle to the other i.e., phase difference between two particles can have any value between 0 and 2	At any instant the phase of all particles between two successive nodes is the same, but phase of particles on one side of a node is opposite to the phase of particles on the other side of the node, i.e, phase difference between any two particles can be either 0 or
4	In these wave, at no instant all the particles of the medium pass through their mean positions simultaneously.	In these waves all particles of the medium pass through their mean position simultaneously twice in each time period.
5	These waves transmit energy in the medium.	These waves do not transmit energy in the medium.

SOLVED EXAMPLE

Ex.1 Given below are some examples of wave motion. State in each case if the wave motion is transverse, longitudinal or a combination of both:
(a) Motion of a kink in a longitudinal spring produced by displacing one end of the spring sideways.
(b) Waves produced in a cylinder containing a liquid by moving its piston back and forth.
(c) Waves produced by a motorboat sailing in water.
(d) Ultrasonic waves in air produced by a vibrating quartz crystal.

Ans.
(a) Transverse and longitudinal
(b) Longitudinal
(c) Transverse and longitudinal
(d) Longitudinal

Ex.2 A wave travelling along a string is described by, y(x, t) = 0.005 sin (80.0 x – 3.0 t), in which the numerical constants are in SI units (0.005 m, 80.0 rad m⁻¹, and 3.0 rad s⁻¹). Calculate (a) the amplitude, (b) the wavelength, and (c) the period and frequency of the wave. Also, calculate the displacement y of the wave at a distance x = 30.0 cm and time t = 20 s ?

Ans. On comparing this displacement equation with Eq. (15.2), y (x, t) = a sin (kx – ωt), we find
(a) the amplitude of the wave is 0.005 m = 5 mm.
(b) the angular wave number k and angular frequency ω are
$k = 80.0\ m^{-1}$ and $\omega = 3.0\ s^{-1}$
We then relate the wavelength ω to k through Eq.

or $\quad k = \frac{2\pi}{\lambda}$

$$\lambda = \frac{2\pi}{80.0\ m^{-1}} = 7.85\ cm$$

(c) Now we relate T to ? by the relation

$$T = \frac{2\pi}{3.0\ s^{-1}}$$

and frequency, v = 1/T = 0.48 Hz The displacement y at x = 30.0 cm and time t = 20 s is given by
y = (0.005 m) sin (80.0 × 0.3 – 3.0 × 20)
= (0.005 m) sin (–36 + 12π)
= (0.005 m) sin (1.699)
= (0.005 m) sin (970) j 5 mm

Ex.3 A steel wire 0.72 m long hasa mass of 5.0 ×10⁻³ kg. If the wire is undera tension of 60 N, what is the speed of transverse waves on the wire

Ans. Mass per unit length of the wire,

$$\mu = \frac{5.0\times10^{-3}kg}{0.72m}$$
$$= 6.9 \times 10^{-3}\ kg\ m^{-1}$$

Tension, T = 60 N

The speed of wave on the wire is given by

$$\upsilon = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{60N}{6.9\times10^{-3}kgm^{-1}}} = 93ms^{-1}$$

Ex.4 Estimate the speed of sound in air at standard temperature and pressure. The mass of 1 mole of air is 29.0 ×10⁻³ kg.

Ans. We know that 1 mole of any gas occupies 22.4 litres at STP. Therefore, densityof air at STP is :
ρ_0 = (mass of one mole of air)/ (volume of one mole of air at STP) $\frac{29.0\times10^3kg}{22.4\times10^3m^3} = 1.29\ kg\ m^{-3}$

According to Newton's formula for the speedof sound in a medium, we get for the speed of sound in air at STP,

$$\upsilon = \left[\frac{1.01\times10^5Nm^{-2}}{1.29kgm^{-3}}\right]^{1/2} = 280\ m\ s^{-1}$$

Ex.5 A pipe, 30.0 cm long, isopen at both ends. Which harmonic modeof the pipe rasonates a 1.1 kHz source? Willresonance with the same source be observed if one end of the pipe is closed ?Take the speed of sound in air as330 m s⁻¹.

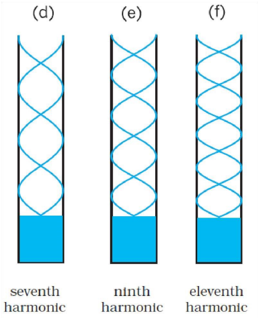
Ans. The first harmonic frequency is givenby

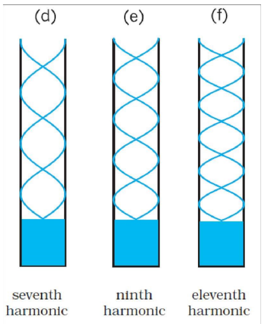
$$\nu_1 = \frac{\upsilon}{\lambda_1} = \frac{\upsilon}{2L} \quad \text{(open pipe)}$$

where L is the length of the pipe. The frequencyof its nth harmonic is:

$$\nu_n = \frac{n\upsilon}{2L}, \text{ for } n = 1, 2, 3, \ldots \text{ (open pipe)}$$

First few modes of an open pipe are shown in fig.





For L = 30.0 cm, v = 330 m s^{-1},

$$\nu_n = \frac{n \times 330(ms^{-1})}{0.6(m)} = 550\ n\ s^{-1}$$

Clearly, a source of frequency 1.1 kHz will resonate at v2, i.e. the **second harmonic**. Now if one end of the pipe is closed (Fig. 15.15), it follows from Eq. (14.50) that the fundamental frequency is

$$\nu_1 = \frac{\upsilon}{\lambda_1} = \frac{\upsilon}{4L}$$ (pipe closed at one end)and only the odd numbered harmonics are present :

$$\nu_3 = \frac{3\upsilon}{4L}, \nu_5 = \frac{5\upsilon}{4L}, \text{ and so on.}$$

For L = 30 cm and v = 330 m s^{-1}, the fundamental frequency of the pipe closed at one end is 275 Hz and the source frequency corresponds to its fourth harmonic. Since this harmonic is not a possible mode, no resonance will be observed with the source, the moment one end is closed.

Ex.6 Two sitar strings A and B playing the note 'Dha' are slightly out of tune and produce beats of frequency 5 Hz. The tension of the string B is slightly increased and the beat frequency is found to decrease to 3 Hz. What is the originalfrequency of B if the frequency of A is427 Hz ?

Ans. Increase in the tension of a string increases its frequency. If the original frequency of B (ν_B) were greater than that of A (ν_A), further increase in ν_B should have resulted in an increase in the beat frequency. But the beat frequency is found to decrease. This shows that

$\nu_B < \nu_A$. Since $\nu_A - \nu_B$ = 5 Hz, and ν_A = 427 Hz, we get ν_B = 422 Hz.

Ex.7 A rocket is moving at a speed of 200 m s^{-1} towards a stationarytarget. While moving, it emits a wave of frequency 1000 Hz. Some of the sound reaching the target gets reflected back to therocket as an echo. Calculate (1) thefrequency of the sound as detected by the target and (2) the frequency of the echo as detected by the rocket.

Ans. (1) The observer is at rest and the source is moving with a speed of 200 m s^{-1}. Since this is comparable with the velocity of sound, 330 m s^{-1}, we must use Eq. and not the approximate Eq. Since the source is approaching a stationary target, υ_0 = 0, and v_s must be replaced by $-\upsilon_s$. Thus, we have

$$\nu = \nu_0\left(1 - \frac{\upsilon_s}{\upsilon}\right)^{-1}$$

ν = 1000 Hz × [1 – 200 m s^{-1} /330 m s^{-1}]$^{-1}$

≅ 2540 Hz

(2) The target is now the source (because it is the source of echo) and the rocket's detector is now the detector or observer (because it detects echo). Thus, v_s = 0 and vo has a positive value. The frequency of the sound emitted by the source (the target) is ν, the frequency intercepted by the target and not ν_0. Therefore, the frequency as registered by the rocket is

$$\nu' = \nu\left(\frac{\upsilon + \upsilon_0}{\upsilon}\right)$$

$$= 2540Hz \times \frac{200ms^{-1} + 330ms^{-1}}{330ms^{-1}}$$

≅ 4080 Hz

(1) $\upsilon = w_0\left(1 + \frac{v_s}{v}\right)^{-1}$

(2) $\upsilon = v_0\left(1 - \frac{v_s}{v}\right)$

Exercise - I UNSOLVED PROBLEMS

Q.1 A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If the transverse jerk is struck at one end of the string, how long does the disturbance take to reach the other and?

Q.2 A stone dropped from the top of a tower of height 300 m high splashes into the water of a pond near the base of the tower. When is the splash heard at the top given that the speed of sound in air is 340 $m\ s^{-1}$?
($g = 9.8\ m\ s^{-2}$).

Q.3 A steel wire has a length of 12.0 m and a mass of 2.10 kg. What should be the tension in the wire so that speed of a transverse wage on the wire equals the speed of sound in dry air at 20 $^{o}C = 343\ m\ s^{-1}$.

Q.4 Use the formula υ = to explain why the speed of sound in air
(a) is independent of pressure,
(b) increases with temperature,
(c) increases with humidity.

Q.5 You have learnt that a travelling wave in one dimension is represented by a function $y = f(x,t)$ where x and t must appear in the combination $x - \upsilon t$ or $x + \upsilon t$, i.e. $y = F(x \pm \upsilon t)$. Is the converse true ? Examine if the following functions for y can possibly represent a travelling wave:
(a) $(x - \upsilon t)^2$
(b) $\log [(x + \upsilon t)/x_o]$
(c) $\exp [-(x + \upsilon t)/x_o]$
(d) $1/(x + \upsilon t)$

Q.6 A bat emits ultrasonic sound of frequency 1000 kHz in air. If the sound meets a water surface, what is the wavelength of (a) the reflected sound, (b) the transmitted sound? Speed of sound in air is 340 $m\ s^{-1}$ and in water 1486 $m\ s^{-1}$.

Q.7 A hospital uses an ultrasonic scanner to locate tumours in a tissue. What is the wavelength os found in the tissue in which the speed of sound is 1.7 km s^{-1} ? The operating frequency of the scanner is 4.2 MHz.

Q.8 A transverse harmonic wave on a string is described by y (x, t) = 3.0 sin
$(36\ t + 0.018\ x + \pi/4)$, where x and y are in cm. and t in s. The positive direction of x is from left to right.
(a) Is this a travelling wave or a stationary wave ?
(b) What are its amplitude and frequency ?
(c) What is the initial phase at the origin ?
(d) What is the least distance between two successive crests in the wave ?

Q.9 For the wave described in Exercise 15.8, plot the displacement (y) versus (t) graphs for x=0, 2 and 4 cm. What are the shape of these graphs ? In which as pacts does the oscillatory motion in traveling wave differ from one point to another : amplitude, frequency of phase ?

Q.10 For the travelling harmonic wave
$y(x,t) = 2.0 \cos 2\pi\ (10t - 0.0080\ x + 0.35)$
where x and y are in cm and t in s. Calculate tha phase difference between oscillatory motion of two points separated by a distance of
(a) 4 m (b) 0.5 m, (c) $\lambda/2$, (d) $3\lambda/4$

Q.11 The transverse displacement of string (clamped at its both ends) is given by
$y(x,t) = 0.06 \sin \cos (120\pi t)$
where and y are in m and t in s. The length os the string is 1.5 m and its mass is 3.0×10^{-2} kg.
Answer the following :
(a) Does the function represent a travelling wave or a stationary wave?
(b) Interpret the wave as a superposition of two waves travelling in opposite directions. What are the wavelength, frequency, and speed of each wave ?
(d) Determine the tension in the string.

Q.12 (i) For the wave on a string described in Exercise 15.11, do all the points on the string oscillate with the same (a) frequency, (b) phases, (c) amplitude ? Explain your answers. (ii) What is the amplitude of a point 0.375 m away from one end ?

Q.13 Given below are some functions of x and t to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represent (i) a travelling wave, (ii) a stationary wave or (iii) none at all:
(a) y = 2 cos (3x) sin (10t) (b) y =
(c) y = 3 sin (5x −0.5t) + cos (5x − 0.5t)
(d) y = cosx sin t + cos 2x sin 2t

Q.14 A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is 3.5 x 10-2 kg and its linear density is 4.0 x 10-2 kg m-1. What is (a) the speed of a transverse wave on the string, and (b) the tension in the string?

Q.15 A meter- long tube open at one end, with a movable piston at the other end, shows resonance with a fixed frequency source (a tuning fork of frequency 340 Hz) when the tube length is 25.5 cm or 79.3 cm. Estimate the speed of sound in air at the temperature of the experiment. The edge effects may be neglected.

Q.16 A steel rod 100 cm long is clamped at its middle. The fundamental frequency of longitudinal vibrations of the rod are given tube 2.53 kHz. What is the speed of sound in steel ?

Q.17 A pipe 20 cm long is closed at one end. Which harmonic mode of the pipe is resonantly excited by a 430 Hz source ? Will the same source be in resonance with the pipe if both ends are open ? (speed of sound in air is 340 m s)

Q.18 Two sitar string A and B playing the note 'Ga' are slightly out of tune and produce beats of frequency 6 Hz. The tension in the string A is slightly reduced and the beat frequency is found to reduce to 3 Hz. If the original frequency of A is 324 Hz, what is the frequency of B ?

Q.19 Explain why (or how):
(a) in a sound wave, a displacement node is pressure antinode and vice versa,
(b) bats can ascertain distances, directions, nature, and sizes of the obstacles without any "eyes",
(c) a violin note and sitar note may have the same frequency, yet we can distinguish between the two notes,
(d) solids can supports both longitudinal and transverse wave, but only longitudinal waves can propagate in gases, and
(e) the shape of a pulse gets distorted during propagation in a dispersive medium.

Q.20 A train, standing at the outer signal of a railway station blows a whistle of frequency 400 Hz in still air.
(i) What is the frequency of the whistle for a platform observer from the platform with a speed of 10 m s^{-1}?
(ii) What is the speed of sound in each case ? The speed of sound in still air can be taken as 340 M s^{-1}.

Q.21 A train , standing in a station-yard, blows a whistle of frequency 400 Hz in still air. The wind starts blowing in the direction from the yard to the station with at a speed of 10 m s^{-1}. What are the frequency, wavelength, and speed of sound for an observer standing on the station's platform? Is the situation exactly identical to the case when the air is still and the observer runs towards the yard at a speed of 10 m s^{-1}? The speed of sound in still air can be taken as 340 m s^{-1}.t

Q.22 A travelling harmonic wave on a string is described by

$$y(x,t) = 7.5 \sin (0.0050x + 12t+\pi/4)$$

(a) what are the displacement and velocity os oscillation of a point at x = 1 cm, and t = 1 cm and t = 1 s ? Is this velocity equal to the velocity of wave propagation?
(b) Locate the points of the sting which have the same transverse displacements and velocity as the x =1 cm point at t = 2 s, 5 s and 11 s.

Q.23 A narrow sound pulse (for example, a short pip by a whistle) is sent across a medium. (a) Does the pulse have a definite (i) frequency, (ii) wavelength, (iii) speed of propagation? (b) If the pulse rate is 1 after every 20 s, (that is the whistle is blown for a split of second after every 20 s), is the frequency os the note produced by the whistle equal to 1.20 or 0.05 Hz ?

Q.24 One end of a long string of linear mass density 8.0×10^{-3} kg m^{-1} is connected to an electrically driven tuning fork of frequency 256 Hz. The other end passes over a pulley and is tied to a pan containing a mass of 90 kg. The pulley end absorbs all the incoming energy so that reflected waves at this end have negligible amplitude. At t = 0, the left end (fork end) of the string x = 0 has zero transverse displacement (y = 0) and is moving along positive y-direction. The amplitude of the wave is 5.0 mm. Write down the transverse displacement y as function of x and t that describes the wave on the string.

Q.25 A SONAR system fixed in a submarine operates at a frequency 40.0 kHz. An enemy submarine moves towards the SONAR with a speed of 360 km h^{-1}. What is the frequency of sound reflected by the submarine ? Take the speed of sound in water to be 1450 m s^{-1}.

Q.26 Earthquakes generate sound waves inside the earth. Unlike a gas, the earth can experience both transverse (S) and logitudinal (P) sound waves. Typically the speed os S wave is about 4.0 km s-1, and that of P wave is 8.0 km s-1. A seismograph records P and S waves from an earthquake. The first P wave arrives 4 min before the first S wave. Assuming the waves travel in straight line, how far away does the earthquake occur?

Q.27 A bat is flitting about in a cave, navigating via ultrasonic bleeps. Assume that the sound emission frequency of the bat is 40 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.03 times the speed of sound in air. What frequency does the bat hear reflected off the wall?