INTRODUCTION TO TRIGONOMETRY

TRIGONOMETRIC IDENTITIES

Trigonometric Identities :

(i) $\sin^2\theta + \cos^2\theta = 1$

(A) $\sin^2\theta = 1 - \cos^2\theta$

(B) $\cos^2\theta = 1 - \sin^2\theta$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(A) $\sec^2\theta - 1 = \tan^2\theta$

(B) $\sec^2\theta - \tan^2\theta = 1$

(C) $\tan^2\theta - \sec^2\theta = -1$

(iii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

(A) $\operatorname{cosec}^2\theta - 1 = \cot^2\theta$

(B) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

(C) $\cot^2\theta - \operatorname{cosec}^2\theta = -1$

Ex.1 Prove the following trigonometric identities :

(i) $(1 - \sin^2\theta)\sec^2\theta = 1$

(ii) $\cos^2\theta\,(1 + \tan^2\theta) = 1$

Sol. **(i)** We have,

LHS = $(1 - \sin^2\theta)\sec^2\theta = \cos^2\theta\sec^2\theta$

$[\because 1 - \sin^2\theta = \cos^2\theta]$

$= \cos^2\theta.\left(\frac{1}{\cos^2\theta}\right) \quad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$

$= 1 = \text{RHS}$

(ii) We have,

LHS $= \cos^2\theta\ (1 + \tan^2\theta)$

$= \cos^2\theta\ .\ \sec^2\theta$

$[\because\ 1 + \tan^2\theta = \sec^2\theta]$

$= \cos^2\theta . \left(\frac{1}{\cos^2\theta}\right)\ \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$

Ex.2 Prove the following trigonometric identities :

(i) $\frac{\sin\theta}{1-\cos\theta} = \text{cosec}\theta + \cot\theta$

(ii) $\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\sec\theta+1}{\sec\theta-1}$

Sol. **(i)** We have,

LHS $= \frac{\sin\theta}{(1-\cos\theta)} \times \frac{(1+\cos\theta)}{(1+\cos\theta)}$

[Multiplying numerator and

denominator by $(1 + \cos\theta)$]

$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$

$[\because\ 1 - \cos^2\theta = \sin^2\theta]$

$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$

$= \text{cosec}\theta + \cot\theta\ = \text{RHS}$

$$\left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta \text{ and } \frac{\cos\theta}{\sin\theta} = \cot\theta\right]$$

(ii) We have,

$$\text{LHS} = \frac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} + \sin\theta}{\frac{\sin\theta}{\cos\theta} - \sin\theta} = \frac{\sin\theta\left(\frac{1}{\cos\theta} + 1\right)}{\sin\theta\left(\frac{1}{\cos\theta} - 1\right)}$$

$$\frac{\frac{1}{\cos\theta} + 1}{\frac{1}{\cos\theta} - 1} = \frac{\sec\theta + 1}{\sec\theta - 1} = \text{RHS}$$

Ex.3 Prove the following identities :

(i) $(\sin\theta + \text{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = 7 + \tan^2\theta + \cot^2\theta$

(ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \text{cosec}\theta)^2 = (1 + \sec\theta\,\text{cosec}\theta)^2$

(iii) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

Sol. **(i)** We have,

LHS $= (\sin\theta + \text{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$= (\sin^2\theta + \text{cosec}^2\theta + 2\sin\theta\,\text{cosec}\theta)\ (\cos^2\theta + \sec^2\theta + 2\cos\theta\,\sec\theta)$

$$= \left(\sin^2\theta + \text{cosec}^2\theta + 2\sin\theta.\frac{1}{\sin\theta}\right) + \left(\cos^2\theta + \sec^2\theta + 2\cos\theta.\frac{1}{\cos\theta}\right)$$

$= (\sin^2\theta + \text{cosec}^2\theta + 2) + (\cos^2\theta + \sec^2\theta + 2)$

$= \sin^2\theta + \cos^2\theta + \text{cosec}^2\,\theta + \sec^2\theta + 4$

$= 1 + (1 + \cot^2\theta) + (1 + \tan^2\theta) + 4$

[$\because$ $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$, $\sec^2\theta = 1 + \tan^2\theta$]

$= 7 + \tan^2\theta + \cot^2\theta$ = RHS.

(ii) We have,

LHS = $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$$= \left(\sin\theta + \frac{1}{\cos\theta}\right)^2 + \left(\cos\theta + \frac{1}{\sin\theta}\right)^2$$

$$= \sin^2\theta + \frac{1}{\cos^2\theta} + \frac{2\sin\theta}{\cos\theta} + \cos^2\theta + \frac{1}{\sin^2\theta} + \frac{2\cos\theta}{\sin\theta}$$

$$= (\sin^2\theta + \cos^2\theta) + \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) + 2\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= (\sin^2\theta + \cos^2\theta) + \left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}\right) + \frac{2(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta}$$

$$= 1 + \frac{1}{\sin^2\theta\cos^2\theta} + \frac{2}{\sin\theta\cos\theta}$$

$$= \left(1 + \frac{1}{\sin\theta\cos\theta}\right)^2 = (1 + \sec\theta\operatorname{cosec}\theta)^2 = \text{RHS}$$

(iii) We have, LHS = $\sec^4\theta - \sec^2\theta$

$= \sec^2\theta(\sec^2\theta - 1) = (1 + \tan^2\theta)(1 + \tan^2\theta - 1)$

[$\because$ $\sec^2\theta = 1 + \tan^2\theta$]

$= (1 + \tan^2\theta)\tan^2\theta = \tan^2\theta + \tan^4\theta$ = RHS.

Ex.4 Prove the following identities :

(i) $\cos^4 A - \cos^2 A = \sin^4 A - \sin^2 A$

(ii) $\cot^4 A - 1 = \text{cosec}^4 A - 2\text{cosec}^2 A$

(iii) $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A \cos^2 A$.

Sol. **(i)** We have,

$\text{LHS} = \cos^4 A - \cos^2 A = \cos^2 A (\cos^2 A - 1)$

$= -\cos^2 A (1 - \cos^2 A) = -\cos^2 A \sin^2 A$

$= -(1 - \sin^2 A) \sin^2 A = -\sin^2 A + \sin^4 A$

$= \sin^4 A - \sin^2 A = \text{RHS}$

(ii) We have,

$\text{LHS} = \cot^4 A - 1 = (\text{cosec}^2 A - 1)^2 - 1$

[$\because \cot^2 A = \text{cosec}^2 A - 1$

$\therefore \cot^4 A = (\text{cosec}^2 A - 1)^2$]

$= \text{cosec}^4 A - 2\,\text{cosec}^2 A + 1 - 1$

$= \text{cosec}^4 A - 2\,\text{cosec}^2 A = \text{RHS}$

(iii) We have,

$\text{LHS} = \sin^6 A + \cos^6 A = (\sin^2 A)^3 + (\cos^2 A)^3$

$= (\sin^2 A + \cos^2 A) \{(\sin^2 A)^2 + (\cos^2 A)^2 - \sin^2 A \cos^2 A)\}$

[$\because a^3 + b^3 = (a + b)(a^2 - ab + b^2)$]

$=\{(\sin^2 A)^2 + (\cos^2 A)^2 + 2 \sin^2 A \cos^2 A - \sin^2 A \cos^2 A\}$

$= [(\sin^2 A + \cos^2 A)^2 - 3 \sin^2 A \cos^2 A]$

$= 1 - 3 \sin^2 A \cos^2 A = \text{RHS}$