TRIANGLES

CRITERIA FOR SIMILARITY OF TRIANGLES

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(i) (AAA Similarity) If two triangles are equiangular, then they are similar.

(ii) (SSS Similarity) If the corresponding sides of two triangles are proportional, then they are similar.

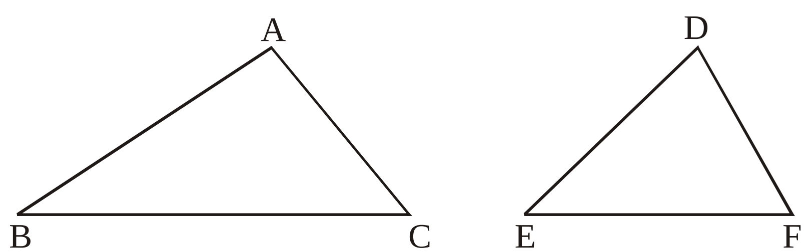
(iii) (SAS Similarity) If in two triangle's one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

ANGLE-ANGLE-ANGLE SIMILARITY (AAA-SIMILARITY) (THEOREM) :

Statement :

In two triangles, if the corresponding angles are equal then their corresponding sides are in the same ratio (or proportion) and hence two triangles are similar.

OR, Two equiangular triangles are similar.



Given : DABC and DDEF are equiangular.

Hence, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$

To prove : $\Delta ABC \sim \Delta DEF$

Proof : Here, ΔABC and ΔDEF are equiangular,

i.e., $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ .....(1)

Three cases arises for sides AB of ΔABC and DE of ΔDEF

(i) AB = DE (ii) AB > DE (iii) AB < DE

Case (1) : When AB = DE

Proof : In ΔABC and ΔDEF

∠A and ∠D (Given)

AB = DE (Given)

∠B and ∠E (Given)

Then by ASA rule of congruence, ΔABC ΔDEF

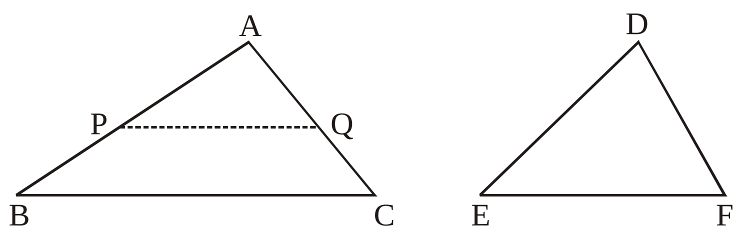
Therefore BC = EF, AC = DE, AB = DE

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{DF}$$

ΔABC ~ ΔDEF

Case (2) : When AB > DE

Construction : As in figure taking the point P and Q on side AB and AC such that AP = DE and AQ = DF.



Proof : In DAPQ and DDEF

AP = DE (By Construction)

AQ = DF (By Construction)

∠A = ∠D (Given)

Therefore by Side-Angle-Side Rule for congruency

ΔAPQ ΔDEF

So, ∠APQ = ∠E .....(1)

But ∠B and ∠E (Given) .....(2)

⇒ ∠APQ = ∠B, which is corresponding angle

Consequently, PQ || BC

Hence $\frac{AP}{AB}=\frac{AQ}{AC}$ (By Basic Proportionality Theorem)

$$\frac{AP}{AQ}=\frac{AB}{AC}$$ .....(3)

Also, $\frac{AP}{DE}=\frac{AQ}{DF}$ (By Construction)

$\Rightarrow$ $\frac{AP}{AQ}=\frac{DE}{DF}$ .....(4)

From (3) and (4),

$\Rightarrow$ $\frac{AB}{AC}=\frac{DE}{DF}$ .....(5)

$\Rightarrow$ $\frac{AB}{DE}=\frac{AC}{DF}$

Similarly, $\frac{AB}{DE}=\frac{BC}{EF}$ .....(6)

From (5) and (6), we get

$$\frac{AB}{DE}=\frac{AC}{DF}=\frac{BC}{EF}$$

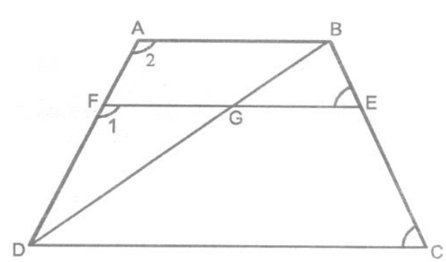
Hence DABC ~ DDEF

Case (3) : When AB < DE, proof is the same as for case (2).

COROLLARY : ANGLE-ANGLE SIMILARITY (AA SIMILARITY)

If two angles of one triangle are respectively equal to two angles of another triangle than the two triangles are similar.

Ex.1 In a trapezium ABCD AB||DC and DC = 2AB. EF drawn parallel to AB cuts AD in F and BC in E such that $\frac{BE}{EC}=\frac{3}{4}$. Diagonal DB intersects EF at G. Prove that 7FE = 10AB.



Sol. In ΔDFG and ΔDAB,

$\angle 1 = \angle 2$ [Corresponding $\angle$s $\therefore$ AB || FG]

$\angle$FDG = $\angle$ADB [Common]

$\therefore \quad \Delta DFG \sim \Delta DAB$ [By AA rule of similarity]

$\therefore \quad \frac{DF}{DA}=\frac{FG}{AB}$ .....(i)

Again in trapezium ABCD

EF||AB||DC

$\therefore \quad \frac{AF}{DF}=\frac{BE}{EC}$

$\Rightarrow \quad \frac{AF}{DF}=\frac{3}{4}$ $\left[\because \frac{BE}{EC}=\frac{3}{4} \text{(given)}\right]$

$\Rightarrow \quad \frac{AF}{DF}=1=\frac{3}{4}+1$

$\Rightarrow \quad \frac{AF+DF}{DF}=\frac{7}{4}$

$\Rightarrow \quad \frac{AD}{DF}=\frac{7}{4}$

$\Rightarrow \quad \frac{DF}{AD}=\frac{4}{7}$ ....(ii)

From (i) and (ii), we get

$\frac{FG}{AB}=\frac{4}{7}$ i.e. FG = $\frac{4}{7}AB$ ......(iii)

In ΔBEG and ΔBCD, we have

$\angle BEG = \angle BCD$ [Corresponding angle $\therefore$ EG||CD]

$\angle GBE = \angle DBC$ [Common]

$\therefore \quad \Delta BEG \sim \Delta BCD$ [By AA rule of similarity]

$\therefore \quad \frac{BE}{BC}=\frac{EG}{CD}$

$\therefore \quad \frac{3}{7}=\frac{EG}{CD}$ $\left[\because \frac{BE}{EG}=\frac{3}{7} \text{ i.e. } \frac{EC}{BE}=\frac{4}{3} \Rightarrow \frac{EC+BE}{BE}=\frac{4+3}{3}\right] \Rightarrow \frac{BC}{BE}=\frac{7}{3}$

$\therefore \quad EG = \frac{3}{7}CD = \frac{3}{7}(2AB) \quad [\because CD = 2AB \text{ (given)}]$

$\therefore \quad EG = \frac{6}{7}AB$ .....(iv)

Adding (iii) and (iv), we get

$FG + EG = \frac{4}{7}AB + \frac{6}{7}AB = \frac{10}{7}AB$

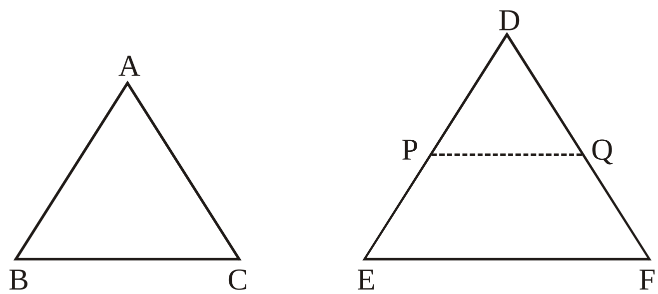
$\Rightarrow \quad EF = \frac{10}{7}AB$ i.e., $7EF = 10AB$ **Hence proved.**

SIDE-SIDE-SIDE SIMILARITY (SSS-SIMILARITY) (THEOREM) :

Statement : If the corresponding sides of two triangles are proportional (i.e. in the same ratio), their corresponding angles are equal and hence the two triangles are similar.

Given : ΔABC and ΔDEF

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} < 1$$



To prove : $\Delta ABC \sim \Delta DEF$

Construction : Taking points P on DE and Q on DF such that DP = AB and DQ = AC, then join PQ.

Proof : In $\Delta ABC \sim \Delta DEF$

$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ .....(1)

$$\frac{AB}{AC}=\frac{DE}{DF}$$ .....(2)

$$\frac{AB}{DP}=\frac{AC}{DQ}$$ (By Construction)

$$\frac{AB}{AC}=\frac{DP}{DQ}$$ .....(3)

From (2) and (3),

$$\frac{DE}{DF}=\frac{DP}{DQ}$$

$$\frac{DP}{DE}=\frac{DQ}{DF}$$

Therefore, by basic proportionality theorem, PQ || EF

So, $\angle DPQ = \angle DEF$ and $\angle DQP = \angle DFE$ (Corresponding angles)

Hence by AA similarity, $\Delta DPQ \sim \angle DEF$ .....(4)

Hence the corresponding sides of similar triangles DDPQ and DDEF are proportional.

i.e. $$\frac{DP}{DE}=\frac{PQ}{EF}$$

$$\frac{AB}{DE}=\frac{PQ}{EF}$$ .....(5)

From (1) and (5),

$$\frac{PQ}{EF}=\frac{BC}{EF}$$

PQ = BC ......(6)

Now, in ΔABC and ΔDPQ

AB = DP (By Construction)

AC = DQ (By Construction)

BC = PQ [From (6)]

So by SSS congruence rule

$\Delta ABC \ \ \Delta DPQ$ .....(7)

From (4) and (7)

$\Delta ABC \sim \Delta DPQ \sim \Delta DEF$

$\Delta ABC \sim \Delta DEF$

SIDE-ANGLE-SIDE SIMILARITY (SAS-SIMILARITY) (THEOREM) :

Statement : If one angle of one triangle is equal to an angle of other triangle and if the side including the angles are proportional, then the two triangles are similar.

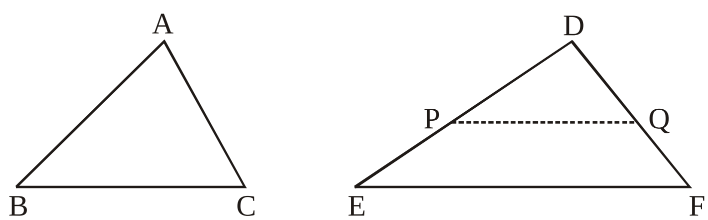
Given : ΔABC and ΔDEF, such that

$$\angle A = \angle D$$

and $$\frac{AB}{DE} = \frac{AC}{DF} < 1$$

To prove : $\Delta ABC \sim \Delta DEF$

Construction : Taking points P and Q on sides DE and DF respectively such that AB = DP andAC=DQ, join PQ.



Proof : In ΔABC and ΔDPQ

$AB = DP$ (By Construction)

$AC = DQ$ (By Construction)

$\angle A = \angle D$ (Given)

By SAS rule of congruence

$\Delta ABC \ \ \Delta DPQ$ .....(1)

$\frac{AB}{DP} = \frac{AC}{DQ}$ (Given) .....(2)

and $\frac{AB}{DE} = \frac{AC}{DF}$ (By Construction) .....(3)

From (2) and (3),

$$\frac{DP}{DQ}=\frac{DE}{DF}$$

$$\frac{DP}{DE}=\frac{DQ}{DF}$$

By converse of basic Proportionality theorem

PQ || EF

So, $\angle DPQ = \angle E$ and $\angle DQP = \angle F$ (Corresponding angles)

Consequently, by AA similarity,

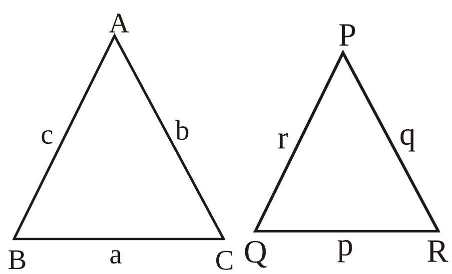
$\Delta DPQ \sim \Delta DEF$ .....(4)

From (1) and (4), we

$\Delta ABC \sim \angle DPQ \sim \angle DEF$

$\Delta ABC \sim \Delta DEF$

Ex. 2 Prove that the ratio of the perimeters of two similar triangles is the same as the ratio of their corresponding sides.



Sol. **Given**: Δ ABC and Δ PQR

BC = a, CA = b, AB = c and QR = p, RP = q, PQ = r

Also $\Delta ABC \sim \Delta PQR$

To prove : $$\frac{a}{p}=\frac{b}{q}=\frac{c}{r}=\frac{a+b+c}{p+q+r}$$

Proof: Since Δ ABC and Δ PQR are similar, there for their corresponding sides are proportional

$\frac{a}{p}=\frac{b}{q}=\frac{c}{r}=k$ (say) .....(i) a = kp, b = kq, c = kr

$$\frac{perimeter\ of\ \Delta ABC}{perimeter\ of\ \Delta PQR} = \frac{a+b+c}{p+q+r} = \frac{kp+kq+kr}{p+q+r}$$

$$= \frac{k(p+q+r)}{(p+q+r)} = k \quad ..... (ii)$$

From (i) and (ii) we get

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \frac{a+b+c}{p+q+r} = \frac{perimeter\ of\ \Delta ABC}{perimeter\ of\ \Delta PQR}$$

Ex.3 $\angle BAC = 90^0$, AD is its bisector. IF DE $\perp$ AC, prove that DE × (AB + AB) = AB × AC.

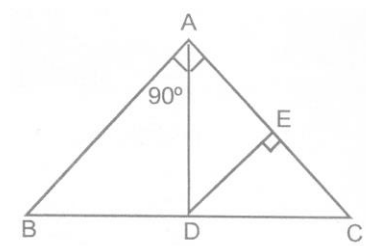
Sol. It is given that AD is the bisector of $\angle A$ of ΔABC.

$$\therefore \quad \frac{AB}{AC} = \frac{BD}{DC}$$

$$\Rightarrow \quad \frac{AB}{AC} + 1 = \frac{BD}{DC} + 1 \quad \text{[Adding 1 on both sides]}$$

$$\Rightarrow \quad \frac{AB+AC}{AC} = \frac{BD+DC}{DC}$$

$$\Rightarrow \quad \frac{AB+AC}{AC} = \frac{BC}{DC} \quad ....(i)$$



In Δ's CDE and CBA, we have

$\angle DCE = \angle BCA$ [Common]

$\angle DEC = \angle BAC$ [Each equal to 90^0]

So, by AA-criterion of similarity

$\Delta\ CDE \sim \Delta\ CBA$

$$\Rightarrow \quad \frac{CD}{CB} = \frac{DE}{BA}$$

$$\Rightarrow \quad \frac{AB}{DE} = \frac{BC}{DC} \quad ....(ii)$$

From (i) and (ii), we have

$$\Rightarrow \quad \frac{AB + AC}{AC} = \frac{AB}{DE}$$

$$\Rightarrow \quad DE \times (AB + AC) = AB \times AC.$$