ARITHMETIC PROGRESSIONS

ARITHMATIC MEAN

Arithmetic Means :

Three quantities are in arithmetical progression, the middle one is said to be the arithmetic mean of the other two.

Thus a is the arithmetic mean between a – d and a + d

(i) Insertion of a single arithmetic mean between a and b

Let A be the arithmetic mean of a and b. Then a, A, b are in A.P.

A – a = b – A

2A = a + b $\Rightarrow A=\frac{a+b}{2}$

(ii) Insertion of n arithmetic means between a and b

Let $A_1, A_2, A_3 \ldots\ldots\ldots A_n$ be n arithmetic means between two quantities a and b.

Then a, $A_1, A_2 \ldots\ldots\ldots A_n$, b is an A.P.

Clearly $b = a_{n+2} = a + [(n + 2) - 1]\, d$

$$\Rightarrow \quad d=\frac{b-a}{n+1}$$

Thus, the n arithmetic means between a and b are as follow:

$A_1 = a + d = a + \frac{b-a}{n+1}$; $A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$; …….. $A_n = a + nd = \frac{n(b-a)}{n+1}$.

$A_n = a+\frac{n(b-a)}{n+1}$

These are required arithmetic means between a and b.

Note: Sum of 'n' arithmetic mean inserted between two numbers a and b is

$S = \frac{n}{2}(a + b)$ where n = number of arithmetic mean.

SUPPOSITION OF TERMS IN A.P.

(i) When no. of terms be odd then we take three terms are as: a – d, a, a + d five terms are as – 2d, a – d, a, a + d, a + 2d

Here we take middle term as 'a' and common difference as 'd'.

(ii) When no. of terms be even then we take 4 term are as : a – 3d, a – d, a + d, a + 3d

6 term are as = a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d

Here we take 'a – d, a + d' as middle terms and common difference as '2d'

Note :

(i) If no. of terms in any series is odd then only one middle term is exist which is $\left(\frac{n+1}{2}\right)^{th}$ term where n is odd.

(ii) If no. of terms in any series is even then middle terms are two which are given by $(n/2)^{th}$ and $\left\{\left(\frac{n}{2}\right)+1\right\}^{th}$ term where n is even.

Properties of Arithmetic Progression :

Property–1: If a constant is added to or subtracted from each term of an AP, then the resulting sequence is also an AP with the same common difference.

Property–2: If each term of a given AP is multiplied or divided by a non-zero constant k, then the resulting sequence is also an AP with common difference kd or d/k respectively. Where d is the common difference of the given AP.

Property–3: In a finite AP the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term

i.e. $a_k + a_{n-(k-1)} = a_1 + a_n$

For all k = 1, 2, 3........(n – 1)

Property–4: Three numbers a, b, c are in AP if 2b = a + c

Property–5: A sequence is an AP if its nth term is a linear expression in n

i.e. $a_n = An + B$ where A, B are constants. In such a case the coefficient of n in a_n is the common difference of the AP.

Property–6: A sequence is an AP if the sum of its first n terms is of the form $An^2 + Bn$ where A, B are constants independent of n. In such a cases the common difference is 2A. i.e. 2 times the coefficient of n^2.

Property–7: If the terms of an AP are chosen at regular intervals then they form an AP.

SOME IMPORTANT RESULTS

(i) The sum of first *n* positive integers

$$S_n = 1 + 2 + 3 + 4 + \ldots\ldots + n$$

$$\left[S_n = \frac{n(n+1)}{2}\right]$$

Note : The sum of first 100 natural numbers is 5050.

(ii) The sum of square of first n positive integers

$$S_n = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$$

$$\left[S_n = \frac{n(n+1)(2n+1)}{6}\right]$$

(iii) The sum of the cubes of first 'n' positive integers

$$S_n = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$$

$$\left[S_n = \left\{\frac{n(n+1)}{2}\right\}^2\right]$$

(iv) $[a_n = S_n - S_{n-1}]$

where a_n = nth term, S_n = sum of n terms

Ex.1 Check whether $a_n = 2n^2 + 1$ is an **A.p.** or not.

Sol. $a_n = 2n^2 + 1$

Then $a_{n+1} = 2(n + 1)^2 + 1$

$\therefore \quad a_{n+1} - a_n = 2(n^2 + 2n + 1) + 1 - 2n^2 - 1$

$= 2n^2 + 4n + 2 + 1 - 2n^2 - 1$

$= 4n + 2$, which is not constant

$\therefore$ The above sequence is not an A.P.

Ex. 2 Insert three arithmetic means between 3 and 19.

Sol. Let A_1, A_2, A_3 be three arithmetic means between 3 and 19. Then

$A_1 = 3 + \frac{1(19-3)}{3+1} = 3 + \frac{16}{4} = 7$

$A_2 = 3 + \frac{2(19-3)}{3+1} = 3 + \frac{32}{4} = 11$

$A_3 = 3 + \frac{3(19-3)}{3+1} = 3 + \frac{48}{4} = 15$

Ex. 3 If a, b, c are in A.P., Prove that b + c, c + a, a + b are in A.P.

Sol. b + c, c + a, a + b will be in A.P.

$(c + a) - (b + c) = (a + b) - (c + a)$

$a - b = b - c$

$2b = a + c$

Thus a, b, c are in A.P.

b + c, c + a, a + b will be in A.P.

Ex.4 The sum of three numbers in A.P. is –3, and their product is 8. Find the numbers.

Sol. Let the numbers be $(a - d), a, (a + d)$. Then,

Sum $= -3 \Rightarrow (a - d) + a\ (a + d) = -3$

$\Rightarrow 3a = -3$

$\Rightarrow a = -1$

Product $= 8$

$\Rightarrow (a - d)\ (a)\ (a + d) = 8$

$\Rightarrow a\ (a^2 - d^2) = 8$

$\Rightarrow (-1)\ (1 - d^2) = 8$

$\Rightarrow d^2 = 9 \Rightarrow d = \pm 3$

If d = 3, the numbers are –4, –1, 2. If d = – 3, the numbers are 2, – 1, –4.

Thus, the numbers are –4, –1, 2, or 2, – 1, – 4.

Ex.5 Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.

Sol. Let the numbers be (a – 3d), (a – d), (a + d),

(a + 3d), Then

Sum = 20

$\Rightarrow$ (a – 3d) + (a – d) + (a + d) + (a + 3d) = 20

$\Rightarrow$ 4a = 20

$\Rightarrow$ a = 5

Sum of the squares = 120

$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120$

$\Rightarrow 4a^2 + 20d^2 = 120$

$\Rightarrow a^2 + 5d^2 = 30$

$\Rightarrow 25 + 5d^2 = 30$ [$\because$ a = 5]

$\Rightarrow 5d^2 = 5 \quad \Rightarrow d = \pm 1$

If d = 1, then the numbers are 2, 4, 6, 8.

If d = – 1, then the numbers are 8, 6, 4, 2. Thus, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

Ex. 6 If a^2, b^2, c^2 are in A.P. then prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

Sol. $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$

$or \frac{a}{b+c}+1, \frac{b}{c+a}+1, \frac{c}{a+b}+1$ are in A.P.

$or \frac{a+b+c}{b+c}, \frac{b+c+a}{c+a}, \frac{c+a+b}{a+b}$ are in A.P.

$or \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$or \frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$or \frac{(b-a)}{(c+a)(b+c)} = \frac{(c-b)}{(a+b)(c+a)}$$

$$or \frac{(b-a)}{(b+c)} = \frac{(c-b)}{(a+b)}$$

Or $b^2 - a^2 = c^2 - b^2$

$2b^2 = a^2 + c^2$

Thus, a^2, b^2, c^2 are in A.P. $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

Ex.7 Divide 32 into four parts which are in A.P. such that the product of extremes is to the product of means is 7 : 15.

Sol. Let the four parts be (a – 3d), (a – d), (a + d) and (a + 3d). Then,

Sum = 32

$\Rightarrow$ (a – 3d) + (a – d) + (a + d) + (a + 3d) = 32

$\Rightarrow$ 4a = 32 $\Rightarrow$ a = 8

It is given that $\frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{7}{15}$

$\Rightarrow \frac{a^2 - 9d^2}{a^2 - d^2} = \frac{7}{15} \Rightarrow \frac{64 - 9d^2}{64 - d^2} = \frac{7}{15}$

$\Rightarrow$ $128d^2 = 512$

$\Rightarrow$ $d^2 = 4 \Rightarrow d = \pm 2$

Thus, the four parts are a – d, a – d, a + d and a + 3d i.e. 2, 6, 10 and 14.

Ex.8 Find the sum of 20 terms of the A.P. 1, 4, 7, 10, ...

Sol. Let a be the first term and d be the common difference of the given A.P. Then, we have a = 1 and d = 3.

We have to find the sum of 20 terms of the given A.P.

Putting a = 1, d = 3, n = 20 in

$S_n = \frac{n}{2}$ [2a + (n – 1) d], we get

$S_{20} = \frac{20}{2} [2 \times 1 + (20 - 1) \times 3]$

$= 10 \times 59 = 590$

Ex.9 Find the sum of first 30 terms of an A.P. whose second term is 2 and seventh term is 22.

Sol. Let a be the first term and d be the common difference of the given A.P. Then,

$a_2 = 2$ and $a_7 = 22$

$\Rightarrow$ a + d = 2 and a + 6d = 22

Solving these two equations, we get

a = – 2 and d = 4.

$S_n = \frac{n}{2} [2a + (n - 1) d]$

$\therefore\ S_{30} = \frac{30}{2} [2 \times (-2) + (30 - 1) \times 4]$

$\Rightarrow 15 (-4 + 116) = 15 \times 112$

$= 1680$

Hence, the sum of first 30 terms is 1680

Ex. 10 If the sum of m terms of an AP is the same as the sum of its n terms, show that the sum of its (m + n) terms is zero.

Sol. Let 'a' be the first term and 'd' be the common difference of the given AP then

$S_m = S_n$

$\frac{m}{2} [2a + (m - 1)d] = \frac{n}{2} [2a + (n - 1)d]$

$m[2a + (m - 1)d] = n[2a + (n - 1)d]$

$2am + m(m - 1)d = 2an + n(n - 1)d$

$2a(m - n) + \{m(m-1) - n(n - 1)\}d = 0$

$2a(m - n) + \{(m^2 - n^2) - (m - n)\}d = 0$

$(m - n)\ [2a + (m + n - 1)d] = 0$

$2a + (m + n - 1)d = 0$ .....(i) $[\therefore\ m - n \neq 0]$

Now, $S_{m+n} = \dfrac{m+n}{2}\ [2a + (m + n - 1)d] = \dfrac{m+n}{2} \times 0 = 0$