POLYNOMIALS

RELATION BETWEEN ZEROS OF POLYNOMIALS

RELATIONSHIP BETWEEN ZEROS AND COEFFICIENTS OF A QUADRATIC POLYNOMIAL :

Let α and β be the zeros of a quadratic polynomial f(x) = $ax^2 + bx + c$. By facto r theorem $(x-\alpha)$ and $(x-\beta)$ are the factors of f(x).

$f(x) = k(x-\alpha)(x-\beta)$ are the factors of f(x)

$\Rightarrow \quad ax^2 + bx + c = k\{x^2 - (\alpha+\beta)x+\alpha\beta\}$

$\Rightarrow \quad ax^2 + bx + c = kx^2 - k(\alpha+\beta)x+k\alpha\beta$

Comparing the coefficients of x^2, x and constant terms on both sides, we get

a = k, b = - k $(\alpha+\beta)$ and $k\alpha\beta$

$\Rightarrow \quad \alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

$\Rightarrow \quad \alpha+\beta=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$ and $\alpha\beta=\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

Hence,

Sum of the zeros $=-\frac{b}{a}=-\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of the zeros $=\frac{c}{a}=\frac{\text{Constant term}}{\text{Coefficient of } x^2}$

REMARKS:

If α and β are the zeros of a quadratic polynomial f (x). The, the polynomial f (x) is given by

$F(x) = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$

Or $f(x) = k\ \{ x^2 - (\text{Sum of the zeros})\ x + \text{Product of the zeros} \}$

Ex.1 Find the zeros of the quadratic polynomial $f(x) = x^2 - 2x - 8$ and verify and the relationship between the zeros and their coefficients.

Sol. $f(x) = x^2 - 2x - 8$

$\Rightarrow f(x) = x^2 - 4x + 2x - 8$

$\Rightarrow f(x) = x(x - 4) + 2(x - 4)]$

$\Rightarrow f(x) = (x - 4)(x + 2)$

Zeros of f(x) are given by $f(x) = 0$

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow (x - 4)(x + 2) = 0$

$\Rightarrow x = 4$ or $x = -2$

So, $\alpha = 4$ and $\beta = -2$

$\therefore$ sum of zeros $\alpha + \beta = 4 - 2 = 2$

Also, sum of zeros $= -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = \dfrac{-(-2)}{1} = 2$

So, sum of zeros $= \alpha + \beta = -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Now, product of zeros $= \alpha\beta = (4)(-2) = -8$

Also, product of zeros $= \dfrac{\text{Constant term}}{\text{Coefficient of } x^2} = \dfrac{-8}{1} = -8$

$\therefore$ Product of zeros $= \dfrac{\text{Constant term}}{\text{Coefficient of } x^2} = \alpha\beta.$

Ex.2 Find a quadratic polynomial whose zeros are $5+\sqrt{2}$ and $5-\sqrt{2}$

Sol. Given $\alpha=5+\sqrt{2}, \beta=5-\sqrt{2}$

$\therefore$ f(x) = $k\{x^2 - x(\alpha+\beta)+\alpha\beta\}$

Here, $\alpha+\beta=5+\sqrt{2}+5-\sqrt{2}=10$

and $\alpha\beta=(5+\sqrt{2})(5-\sqrt{2})$

$= 25 - 2 = 23$

$\therefore$ f(x) = k $\{x^2 - 10x + 23\}$, where, k is any non-zero real number.

Ex.3 Sum of product of zeros of quadratic polynomial are 5 and 17 respectively. Find the polynomial.

Sol. Given : Sum of zeros = 5 and product of zeros = 17

So, quadratic polynomial is given by

$\Rightarrow$ f(x) = k $\{x^2$ - x(sum of zeros) + product of zeros$\}$

$\Rightarrow$ f(x) = $k\{x^2 - 5x + 17\}$, where, k is any non-zero real number,

RELATIONSHIP BETWEEN ZEROS AND COEFFICIENTS OF A CUBIC POLYNOMIAL :

Let α, β, γ be the zeros of a cubic polynomial f(x) = $ax^3 + bx^2 + cx + d$, $a\neq 0$ Then, by factor theorem, $a-\alpha, x-\beta$ and $x-\gamma$ are factors of f(x). Also, f(x) being a cubic polynomial cannot have more than three linear factors.

$\therefore$ f(x) = $k(x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow$ $ax^3 + bx^2 + cx + d = k(x-\alpha)(x-\beta)(x-\gamma)$

$\Rightarrow$ $ax^3 + bx^2 + cx + d = k\{x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma\}$

$\Rightarrow$ $ax^3 + bx^2 + cx + d = k x^3 - k(\alpha+\beta+\gamma)x^2 + k(\alpha\beta+\beta\gamma+\gamma\alpha)x - k\alpha\beta\gamma$

Comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

a = k, b = - k $(\alpha+\beta+\gamma), c = (\alpha\beta+\beta\gamma+\gamma\alpha)$ and $d = -k(\alpha\beta\gamma)$

$\Rightarrow \quad \alpha+\beta+\gamma = -\frac{b}{a}$

$\Rightarrow \quad \alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$

And, $\quad \alpha\beta\gamma = -\frac{d}{a}$

$\Rightarrow$ Sum of the zeros $= -\frac{b}{a} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$

$\Rightarrow$ Sum of the products of the zeros taken two at a time $= \frac{c}{a} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$

$\Rightarrow$ Product of the zeros $= -\frac{d}{a} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$

REMARKS:

Cubic polynomial having α, β *and* γ as its zeros is given by

f(x) = k $(x-\alpha)(x-\beta)(x-\gamma)$

f(x) = k $\{x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma\}$ where k is any non- zero real number.

Ex.4 Verify that $\frac{1}{2}, 1 - 2$ are zeros of cubic polynomial $2x^3 + x^2 - 5x + 2$. Also verify the relationship between, the zeros and their coefficients.

Sol. $f(x) = 2x^3 + x^2 - 5x + 2$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2$$

f(1) = 2()^3 + (1)^2 5(1) + 2 = 2 + 1 - 5 + 2 = 0.

f(-2) = 2(-2)^3 + (-2)^2 - 5(-2) + 2 = - 16 + 4 + 10 + 2 = 0.

Let $\alpha=\frac{1}{2}, \beta=1$ and $\gamma=-2$

Now, Sum of zeros $=\alpha+\beta+\gamma$

$$=\frac{1}{2}+1-2 \quad =-\frac{1}{2}$$

Also, sum of zeros $=-\frac{(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3}=-\frac{1}{2}$

So, sum of zeros $=\alpha+\beta+\gamma=-\frac{(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3}$

Sum of product of zeros taken two at a time $=\alpha\beta+\beta\gamma+\gamma\alpha$

$$=\frac{1}{2}\times1+1\times(-2)+(-2)\times\frac{1}{2}$$

$$=-\frac{5}{2}$$

Also, $\beta\beta+\beta\gamma+\gamma\alpha=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}=\frac{-5}{2}$

So, sum of product of zeros taken two at a time = $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$

Now, Product of zeros = $\alpha\beta\gamma \quad =\left(\frac{1}{2}\right)(1)(-2)=-1$

Also, product of zeros = $\frac{\text{Constant term}}{\text{Coefficient of } x^3}=\frac{-2}{2}=-1$

Product zeros = $\alpha\beta\gamma=-\frac{\text{Constant term}}{\text{Coefficient of } x^3}$

Ex.5 Find a polynomial with the sum, sum of the product of its zeros taken two at a time, and product its zeros as 3, -1 and -3 respectively.

Sol. Given $\alpha+\beta+\gamma=3, \alpha\beta+\beta\gamma+\gamma\alpha=-1$ and $\alpha\beta\gamma=-3$

So, polynomial f(x) = k {x^3 - x^2 $(\alpha+\beta+\gamma)+x(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma$}

f(x) = k {x^3 - $3x^2$ - x + 3}, where k is any non-zero real number.

Some useful relations involving a and b :

1. $a^2 + b^2 = (a + b)^2 – 2ab$
2. $(a – b)^2 = (a + b)^2 – 4ab$
3. $a^2 – b^2 = (a + b)\ (a – b) = (a + b)$

   $\sqrt{(\alpha+\beta)^2-4\alpha\beta}$
4. $a^3 + b^3 = (a + b)^3 – 3ab\ (a + b)$
5. $a^3 – b^3 = (a – b)^3 + 3ab\ (a – b)$
6. $a^4 + b^4 = [(a + b)^2 – 2ab]^2 –2(ab)^2$
7. $a^4 – b^4 = (a^2 + b^2)\ (a^2 – b^2)$ then use (1) and (3)

Ex.6 If a and b are the zeroes of the polynomial $ax^2 + bx + c$. Find the value of

(i) a – b (ii) $a^2 + b^2$.

Sol. Since a and b are the zeroes of the polynomial $ax^2 + bx + c$.

$a + b = -\frac{b}{a}$; $ab = \frac{c}{a}$

(i) $(a – b)^2 = (a + b)^2 – 4ab$

$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2-4ac}{a^2}$

$a – b = \frac{\sqrt{b^2-4ac}}{a}$

(ii) $a^2 + b^2 = a^2 + b^2 + 2ab - 2ab$

$= (a + b)^2 - 2ab$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2 - 2ac}{a^2}$

TO FORM A QUADRATIC POLYNOMIAL WITH THE GIVEN ZEROES

Let zeroes of a quadratic polynomial be α *and* β .

$x = \alpha$ $\qquad$ $x = \beta$

$x - \alpha = 0$ $\qquad$ $x - \beta = 0$

The obviously the quadratic polynomial is

$(x - \alpha)(x - \beta)$

i.e. $x^2 - (\alpha + \beta)x + \alpha\beta$

x^2 – (Sum of the zeroes) x + Product of the zeroes

Ex.7 Form the quadratic polynomial whose zeroes are 4 and 6.

Sol. Sum of the zeroes = 4 + 6 = 10

Product of the zeroes = 4 × 6 = 24

Hence the polynomial formed

$= x^2$ – (sum of zeroes) x + Product of zeroes

$= x^2 - 10x + 24$

Ex.8 Form the quadratic polynomial whose zeroes are –3, 5.

Sol. Here, zeroes are – 3 and 5.

Sum of the zeroes = – 3 + 5 = 2

Product of the zeroes = (–3) × 5 = – 15

Hence the polynomial formed = x^2 – (sum of zeroes) x + Product of zeroes

$= x^2 - 2x - 15$

Ex.9 Find a quadratic polynomial whose sum of zeroes and product of zeroes are respectively-

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

Sol. Let the polynomial be $ax^2 + bx + c$ and its zeroes be a and b.

(i) Here, $a + b = \frac{1}{4}$ and $a . b = -1$

Thus the polynomial formed

$= x^2 -$ (Sum of zeroes) x + Product of zeroes

$= x^2 - \left(\frac{1}{4}\right) x - 1 = x^2 - \frac{x}{4} - 1$

The other polynomial are $k\left(x^2 - \frac{x}{4} - 1\right)$

If k = 4, then the polynomial is $4x^2 - x - 4$.

(ii) Here, $a + b = \sqrt{2}$, $ab = \frac{1}{3}$

Thus the polynomial formed

$= x^2 -$ (Sum of zeroes) x + Product of zeroes

$= x^2 - (\sqrt{2})\, x + \frac{1}{3}$ or $x^2 - \sqrt{2}\, x + \frac{1}{3}$

Other polynomial are $k\left(x^2 - \sqrt{2}x + \frac{1}{3}\right)$

If k = 3, then the polynomial is

$3x^2 - 3\sqrt{2}\, x + 1$

(iii) Here, $a + b = 0$ and $a.b = \sqrt{5}$

Thus the polynomial formed

$= x^2 -$ (Sum of zeroes) x + Product of zeroes

$= x^2 - (0)\, x + \sqrt{5} = x^2 + \sqrt{5}$